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Spin, bond and global fluctuation–dissipation relations in the non-equilibrium spherical ferromagnet

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Abstract

We study the out-of-equilibrium dynamics of the spherical ferromagnet after a quench to its critical temperature. We calculate correlation and response functions for spin observables which probe length scales much larger than the lattice spacing but smaller than the system size, and find that the asymptotic fluctuation–dissipation ratio (FDR) X^∞ is the same as for local observables. This is consistent with our earlier results for the Ising model in dimension $d = 1$ and $d = 2$. We also check that bond observables, both local and long range, give the same asymptotic FDR. In the second part of the paper the analysis is extended to global observables, which probe correlations among all N spins. Here, non-Gaussian fluctuations arising from the spherical constraint need to be accounted for, and we develop a systematic expansion in $1/\sqrt{N}$ to do this. Applying this to the global bond observable, i.e. the energy, we find that non-Gaussian corrections change its FDR to a nontrivial value which we calculate exactly for all dimensions $d > 2$. Finally, we consider quenches from magnetized initial states. Here, even the FDR for the global *spin* observable, i.e. the magnetization, is nontrivial. It differs from the one for unmagnetized states even in $d > 4$, signalling the appearance of a distinct dynamical universality class of magnetized critical coarsening. For lower d, the FDR is irrational even to first order in $4 - d$ and $d - 2$, the latter in contrast to recent results for the transverse FDR in the n-vector model.

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1. Introduction

A key insight of statistical mechanics is that *equilibrium* states can be accurately described in terms of only a small number of thermodynamic variables, such as temperature and pressure. For non-equilibrium systems such as glasses no similar simplification exists *a priori*; the

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whole past history of a sample is in principle required to specify its state at a given time. This complexity makes theoretical analysis awkward, and one is led instead to look for a description of non-equilibrium states in terms of a few effective thermodynamic parameters. Much work in recent years has focused on one such parameter, the *effective temperature*. This can be defined on the basis of fluctuation–dissipation (FD) relations between correlation and response functions and has proved to be very fruitful in mean-field systems [1, 2].

The use of FD relations to quantify the out-of-equilibrium dynamics in glassy systems is motivated by the occurrence of *ageing* [3]: the time scale of response to an external perturbation increases with the age (time since preparation) t_w of the system. As a consequence, time translational invariance and the equilibrium fluctuation–dissipation theorem [4] (FDT) relating correlation and response functions break down. To quantify this, one considers the correlation function of two generic observable A and B of a system, defined as

$$C(t, t_w) = \langle A(t)B(t_w)\rangle - \langle A(t)\rangle\langle B(t_w)\rangle. \tag{1.1}$$

The associated (impulse) response function can be defined as

$$R(t, t_w) = \left.\frac{\delta\langle A(t)\rangle}{\delta h_B(t_w)}\right|_{h_B=0}$$

and gives the linear response of A at time t to a small impulse in the field h_B conjugate to B at the earlier time t_w. (The latter is normally thought of as a 'waiting time' since preparation of the system at time 0.) Equivalently, one can work with the susceptibility

$$\chi(t, t_w) = \int_{t_w}^{t} dt'\, R(t, t') \tag{1.2}$$

which encodes the response of $A(t)$ to a small step $h_B(t) = h_B\Theta(t - t_w)$ in the field starting at t_w. In equilibrium, FDT implies that $-\partial_{t_w}\chi(t, t_w) = R(t, t_w) = T^{-1}\partial_{t_w}C(t, t_w)$. Out of equilibrium, the violation of FDT can be measured by an FD ratio (FDR), X, defined through [5, 6]

$$-\partial_{t_w}\chi(t, t_w) = R(t, t_w) = \frac{X(t, t_w)}{T}\partial_{t_w}C(t, t_w). \tag{1.3}$$

This implies that X can be read off from the slope $-X/T$ of a parametric FD plot showing χ versus C, at fixed t and with t_w as the curve parameter. This remains the case if both axes are normalized by the equal-time covariance of A and B, $C(t, t)$, a procedure which is helpful in fixing the scale of the plot in situations where $C(t, t)$ varies significantly with time [7, 8]. In equilibrium, the FD plot is a straight line with slope $-1/T$.

In mean-field spin glasses [1, 5, 6], one finds that FD plots of autocorrelation and response of local spins and similar observables approach a limiting shape for large t. This is typically composed of two straight-line segments. In the first of these one finds $X = 1$, corresponding to quasi-equilibrium dynamics for time differences $t - t_w$ that do not grow with the age of the system. The second line segment has $X < 1$ and reflects the dynamics on ageing time scales, i.e. time differences growing (in the simplest case linearly) with t_w. One can use this to define a non-equilibrium *effective temperature* $T_{eff} = T/X$, which has been shown to have many of the properties of a thermodynamic temperature [1, 5, 6].

How well this physically very attractive mean-field scenario transfers to more realistic non-equilibrium systems with short-range interactions has been a matter of intense research recently [2]. A useful class of systems for studying this question in detail is provided by ferromagnets quenched from high temperature to the critical temperature (see, e.g., [9, 10] and the recent review [11]) or below. The system then coarsens—by the growth of domains with the equilibrium magnetization, for $T < T_c$—and exhibits ageing; in an infinite system equilibrium is never reached. The ageing is clearly related to the growth of a length scale [12] (domain size for $T < T_c$ or correlation length for $T = T_c$), and this makes ferromagnets attractive 'laboratory' systems for understanding general properties of non-equilibrium dynamics. They are of course not completely generic; compared to, e.g., glasses they lack features such as thermal activation over energetic or entropic barriers.

We focus in this paper mostly on ferromagnets quenched to T_c, i.e. on critical coarsening dynamics. Some care is needed in this case with the interpretation of limiting FD plots: while in the mean-field situation X becomes at long times a function of C only, as implied by the existence of a nontrivial limit plot, in critical coarsening X approaches a function of t/t_w [9]. In the interesting regime where t/t_w is finite but >1, time differences $t - t_w = \mathcal{O}(t_w)$ are then large, and, e.g., spin autocorrelation functions have decayed to a small part of their initial value. In the limit $t, t_w \to \infty$, the FD plot then assumes a pseudo-equilibrium shape, with all nontrivial detail compressed into a vanishingly small region around $C = 0$.

The fact that the FDR is a smooth function of t/t_w makes the interpretation of T/X as an effective temperature less obvious than in mean-field spin glasses, where T/X is constant within each time sector ($t - t_w = \mathcal{O}(1)$ versus $t - t_w$ growing with t_w). To eliminate the time dependence one can consider the limit of times that are both large and well separated. This defines an *asymptotic FDR*

$$X^\infty = \lim_{t_w\to\infty} \lim_{t\to\infty} X(t, t_w). \tag{1.4}$$

An important property of this quantity is that it should be *universal* [9, 11] in the sense that its value is the same for different systems falling into the same universality class of critical non-equilibrium dynamics. This makes a study of X^∞ interesting in its own right, even without an interpretation in terms of effective temperatures.

If one nevertheless wants to pursue such an interpretation, the resulting value of the effective temperature T/X^∞ should be the same for all (or at least a large class of) observables A. The observable dependence of X^∞ therefore becomes a key question [7, 8, 10, 13]. Conventionally, much work on non-equilibrium ferromagnets has focused on the local spin autocorrelation function and associated response. An obvious alternative is the long-wavelength analogue, i.e. the correlation function of the fluctuating magnetization. Exact calculations for the Ising chain [10, 14] as well as numerical simulations [10, 15] in dimension $d = 2$ show that the resulting global X^∞ is always identical to the local version. This local–global correspondence, which can also be obtained by field-theoretic arguments [11, 16, 17], arises physically because the long-wavelength Fourier components of the spins are slowest to relax and dominate the long-time behaviour of both local and global quantities.

The local–global correspondence does of course not address the full range of observable dependence of the asymptotic FDR; one might ask about other observables which are linear combinations not of spins but, for example, products of interacting spins. In the critical Ising model in $d = 2$, numerical simulations [10, 15] suggest that even these give the same X^∞, so that an interpretation of T/X^∞ in terms of an effective temperature appears plausible. One of the motivations for the current study was to verify whether this *observable independence* of X^∞ across different types of observables holds in an exactly solvable model, the spherical ferromagnet [18, 19]. In addition, we will study what effect different *initial conditions* have on X^∞. This is motivated by our recent study of Ising models in the classical regime of large d, where critical fluctuations are irrelevant [20]. It turned out that magnetized initial states do produce a different value of X^∞, so that critical coarsening in the presence of a nonzero magnetization is in a new dynamical universality class even though the magnetization does decay to zero at long times.

We begin in section 2 with a brief review of the standard set-up for the dynamics of the spherical model, as used in, e.g., [9]. Fluctuations in an effective Lagrange multiplier enforcing the spherical constraint are neglected, leading to a theory where all spins are Gaussian random variables. In section 3 this is applied to various observables of finite range, by which we mean correlations and responses probing length scales that can be large but remain small compared to the overall system size. For spin observables, we show that the expected equality of X^∞ between local and long-range quantities holds (section 3.1). We check observable independence of X^∞ further by considering bond and spin product observables, in section 3.2 and 3.3, respectively.

The major part of the paper is then devoted to a study of FDRs for *global* observables, with a focus on the energy, i.e. the global bond observable. Because of the weak infinite-range interaction generated by the spherical constraint, such observables behave differently from their long-range analogues in the spherical model. Calculations of correlation and response functions are technically substantially more difficult because Lagrange multiplier fluctuations can no longer be neglected. To account for them, we construct in section 4 a systematic expansion of the dynamically evolving spins in $N^{-1/2}$. This allows us to calculate the leading non-Gaussian corrections that we need for global correlations, as shown for the case of the energy in section 5. After a brief digression to equilibrium dynamics, we evaluate the resulting expressions in section 6 for d above the critical dimension $d_c = 4$ and in section 7 for $d < 4$. Importantly, we will find that in the latter case the asymptotic FDR is different from that for finite-range observables. This means that an effective temperature interpretation of X^∞ is possible at best in a very restricted sense. However, we will find that our results are in agreement with recent renormalization group (RG) calculations near $d = 4$ [13] in the $O(n \to \infty)$-model. This suggests that the non-Gaussian effects captured in global observables are important for linking the spherical model to more realistic systems with only short-range interactions. Finally, we turn in section 8 to critical coarsening starting from magnetized initial conditions. Here, already the global *spin* observable is affected by non-Gaussian corrections. Once these are accounted for, we find $X^\infty = 4/5$ for $d > 4$ as in the Ising case [20]. For $d < 4$, we provide the first exact values of the asymptotic FDR in the presence of a nonzero magnetization; these turn out to be highly nontrivial even to first order in $4 - d$ and $d - 2$. Our results are summarized and discussed in section 9. Technical details are relegated to two appendices.

2. Langevin dynamics and Gaussian theory

We consider the standard spherical model Hamiltonian

$$H = \frac{1}{2}\sum_{(ij)}(S_i - S_j)^2. \tag{2.1}$$

The sum runs over all nearest neighbour (n.n.) pairs on a d-dimensional (hyper-)cubic lattice; the lattice constant is taken as the unit length. At each of the N lattice sites $\mathbf{r}_i$ there is a real-valued spin S_i. The spherical constraint $\sum_i S_i^2 = N$ is imposed, which can be motivated by analogy with Ising spins $S_i = \pm 1$ [18].

The Langevin dynamics for this model can be written as

$$\partial_t S_i = -\frac{\partial H}{\partial S_i} + \xi_i - \frac{1}{N}\sum_k S_k\left(-\frac{\partial H}{\partial S_k} + \xi_k\right)S_i \tag{2.2}$$

with ξ_i Gaussian white noise with zero mean and covariance $\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t - t')$. The last term in (2.2), i.e. the sum over k, enforces the spherical constraint at all times by removing the component of the velocity vector $(\partial_t S_1, \ldots, \partial_t S_N)$ along $(S_1, \ldots, S_N)$. We use here the Stratonovic convention for products like $S_k \xi_k$. This allows the ordinary rules of calculus to be used when evaluating derivatives such as $\partial_t S_i^2$. Physically, it corresponds to the intuitively reasonable scenario where the noise ξ_i is regarded as a smooth random process but with a correlation time much shorter than any other dynamical time scale.

The prefactor of S_i in the last term of (2.2), being an average of N contributions, will have fluctuations of $\mathcal{O}(N^{-1/2})$. Conventionally, one ignores these and approximates the equation of motion as

$$\partial_t S_i = -\frac{\partial H}{\partial S_i} + \xi_i - z(t) S_i \tag{2.3}$$

where $z(t)$ can be viewed as an effective time-dependent Lagrange multiplier implementing the spherical constraint. This approximation works for local quantities, but as we will see can give incorrect results when one considers, e.g., fluctuations of the magnetization or the energy, which involve correlations across the entire system. One can directly see that (2.3) is an approximation from the fact that it corresponds to Langevin dynamics with the effective Hamiltonian $H + \frac{1}{2} z(t) \sum_i S_i^2$. Since the latter is time dependent, this dynamics does not satisfy detailed balance. It is simple to check, on the other hand, that the original equation of motion (2.2) does satisfy detailed balance and leads to the correct equilibrium distribution $P_{\mathrm{eq}}(\{S_i\}) \propto \exp(-\beta H) \delta\big(\sum_i S_i^2 - N\big)$ where $\beta = 1/T$ is the inverse temperature as usual.

The key advantage of the approximation (2.3) is, of course, that the spins are Gaussian random variables at all times as long as the initial condition is of this form. Explicitly, if we define a matrix Ω with $\Omega_{ij} = -1$ for n.n. sites i, j and $\Omega_{ii} = 2d$, the Gaussian equation of motion is

$$\partial_t S_i = -\sum_j \Omega_{ij} S_j - z(t) S_i + \xi_i. \tag{2.4}$$

We review briefly how this is solved (see, e.g., [9] and references therein), since these results form the basis for all later developments. In terms of the Fourier components $S_{\mathbf{q}} = \sum_i S_i \exp(-\mathrm{i}\mathbf{q} \cdot \mathbf{r}_i)$ of the spins, equation (2.4) reads

$$\partial_t S_{\mathbf{q}} = -(\omega_{\mathbf{q}} + z(t)) S_{\mathbf{q}} + \xi_{\mathbf{q}} \tag{2.5}$$

where $\omega_{\mathbf{q}} = 2 \sum_{a=1}^{d} (1 - \cos q_a)$; we mostly write just ω. The Fourier mode response function can be read off as

$$R_{\mathbf{q}}(t, t_{\mathrm{w}}) = \exp\left(-\omega(t - t_{\mathrm{w}}) - \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t'\, z(t')\right) \equiv \sqrt{\frac{g(t_{\mathrm{w}})}{g(t)}}\, \mathrm{e}^{-\omega(t - t_{\mathrm{w}})} \tag{2.6}$$

where

$$g(t) = \exp\left(2 \int_0^t \mathrm{d}t'\, z(t')\right). \tag{2.7}$$

In terms of this, the time dependence of the $S_{\mathbf{q}}$ becomes

$$S_{\mathbf{q}}(t) = R_{\mathbf{q}}(t, 0) S_{\mathbf{q}}(0) + \int_0^t \mathrm{d}t'\, R_{\mathbf{q}}(t, t') \xi_{\mathbf{q}}(t'). \tag{2.8}$$

The equal-time correlator $C_{\mathbf{q}}(t, t) = (1/N)\langle S_{\mathbf{q}}(t) S_{\mathbf{q}}^*(t)\rangle$ follows as

$$C_{\mathbf{q}}(t, t) = C_{\mathbf{q}}(0, 0) R_{\mathbf{q}}^2(t, 0) + 2T \int_0^t \mathrm{d}t'\, R_{\mathbf{q}}^2(t, t') \tag{2.9}$$

$$= \frac{C_{\mathbf{q}}(0, 0)}{g(t)}\, \mathrm{e}^{-2\omega t} + 2T \int_0^t \mathrm{d}t'\, \frac{g(t')}{g(t)}\, \mathrm{e}^{-2\omega(t - t')} \tag{2.10}$$

and we note for later the identity

$$\partial_t C_{\mathbf{q}}(t,t) = 2T - \left(2\omega + \frac{g'(t)}{g(t)}\right) C_{\mathbf{q}}(t,t). \tag{2.11}$$

The two-time correlator $C_{\mathbf{q}}(t,t_w) = (1/N)\langle S_{\mathbf{q}}(t) S^*_{\mathbf{q}}(t_w)\rangle$ can be deduced from the analogue of (2.8) for initial time t_w

$$S_{\mathbf{q}}(t) = R_{\mathbf{q}}(t,t_w) S_{\mathbf{q}}(t_w) + \int_{t_w}^{t} dt'\, R_{\mathbf{q}}(t,t')\xi_{\mathbf{q}}(t') \tag{2.12}$$

as

$$C_{\mathbf{q}}(t,t_w) = R_{\mathbf{q}}(t,t_w) C_{\mathbf{q}}(t_w,t_w). \tag{2.13}$$

The position-dependent correlation and response functions $C_{ij}(t,t_w)$ and $R_{ij}(t,t_w)$ are then just the inverse Fourier transforms of $C_{\mathbf{q}}(t,t_w)$ and $R_{\mathbf{q}}(t,t_w)$, respectively, with $\mathbf{q}$ conjugate to $\mathbf{r}_j - \mathbf{r}_i$.

2.1. The function $g(t)$

The calculations outlined above show that the Gaussian dynamics is fully specified once the function $g(t)$ is known. The latter can be found from the spherical constraint, which imposes $\int (dq) C_{\mathbf{q}}(t,t) = 1$. Here and below, we abbreviate $(dq) \equiv d\mathbf{q}/(2\pi)^d$, where the integral runs over the first Brillouin zone of the hypercubic lattice, i.e. $\mathbf{q} \in [-\pi,\pi]^d$. Using (2.10), this constraint gives an integral equation for $g(t)$:

$$g(t) = \int (dq) C_{\mathbf{q}}(0,0)\, e^{-2\omega t} + 2T \int_0^t dt'\, f(t-t') g(t') \tag{2.14}$$

where

$$f(t) = \int (dq)\, e^{-2\omega t} = [e^{-4t} I_0(4t)]^d \approx (8\pi t)^{-d/2}. \tag{2.15}$$

Here, I_0 denotes a modified Bessel function and the final expression gives the asymptotic behaviour for large t. In terms of Laplace transforms $\hat{g}(s) = \int_0^\infty dt \exp(-st) g(t)$, equation (2.14) then has the solution

$$\hat{g}(s) = \frac{1}{1 - 2T\hat{f}(s)} \int (dq) \frac{C_{\mathbf{q}}(0,0)}{s + 2\omega}. \tag{2.16}$$

With the exception of section 8, in this paper we focus on random initial conditions, $C_{\mathbf{q}}(0,0) = 1$, corresponding to a quench at time $t = 0$ from equilibrium at infinite temperature. In this case, the $\mathbf{q}$-integral in the last equation is just $\hat{f}(s)$, so that

$$\hat{g}(s) = \frac{\hat{f}(s)}{1 - 2T\hat{f}(s)}. \tag{2.17}$$

The asymptotics of the corresponding $g(t)$ are well known; see, e.g., [9, 21]. For T above the critical temperature T_c, which is given by

$$T_c^{-1} = 2\hat{f}(0) = \int (dq) \frac{1}{\omega} \tag{2.18}$$

there is a pole in $\hat{g}(s)$ at $s = 2z_{eq}$. Here, z_{eq} is found from the condition $2T\hat{f}(2z_{eq}) = 1$ or

$$\int (dq) \frac{T}{z_{eq} + \omega} = 1. \tag{2.19}$$

The presence of this pole tells us that $g(t) \sim \exp(2z_{eq} t)$ for long times, implying that the Lagrange multiplier $z(t)$ approaches z_{eq} for $t \to \infty$. Correspondingly, condition (2.19) is just the spherical constraint at equilibrium, bearing in mind that $C^{\rm eq}_{\mathbf{q}}(t,t) = T/(z_{\rm eq}+\omega)$ from (2.5). Because $\omega \approx q^2$ for small $q = |\mathbf{q}|$, the phase space factor in the $\mathbf{q}$-integrals is $(\mathrm{d}q) \sim \mathrm{d}\omega\,\omega^{d/2-1}$ for small q or ω. This shows that $T_{\rm c}$ as given by (2.18) vanishes as $d \to 2$ from above; consequently, we will always restrict ourselves to dimensions d above this lower critical dimension.

At criticality ($T = T_{\rm c}$), $z_{\rm eq}$ vanishes, and $g(t)$ therefore no longer grows exponentially; instead one finds [9, 21]

$$g(t) \sim t^{-\kappa}, \qquad \kappa = \begin{cases} (4-d)/2 & \text{for} \quad 2<d<4 \\ 0 & \text{for} \quad d>4. \end{cases} \tag{2.20}$$

It is this case, of a quench to the critical temperature, that we will concentrate on throughout most of this paper. This is because here the FDR has the most interesting behaviour.

We note briefly that, in principle, $\int(\mathrm{d}q)$ should be written as $(1/N)\sum_{\mathbf{q}}$, with the sum running over all $\mathbf{q}$ whose components are integers in the range $-L/2,\ldots,-1,0,1,\ldots,L/2-1$ (assuming the linear system size L is even) multiplied by an overall factor $2\pi/L$; there are $N = L^d$ such $\mathbf{q}$. When considering continuous functions of $\mathbf{q}$ this sum can be replaced by the integral $\int(\mathrm{d}q)$, and this will almost always be the case in our analysis. Exceptions are situations with a nonzero magnetization, where the wavevector $\mathbf{q} = \mathbf{0}$ is special and has to be treated separately. This is relevant in equilibrium below $T_{\rm c}$, which we discuss briefly in section 5.3, and for non-equilibrium dynamics starting from magnetized initial states (section 8).

2.2. *Long-time scaling of* $C_{\mathbf{q}}$

It will be useful later to have a simplified long-time expression for $C_{\mathbf{q}}(t_{\rm w},t_{\rm w})$ for the case of a critical quench. At zero wavevector, one has

$$C_{\mathbf{0}}(t_{\rm w},t_{\rm w}) = \frac{1}{g(t_{\rm w})}\left(1 + 2T_{\rm c}\int_0^{t_{\rm w}} \mathrm{d}t'\,g(t')\right) \approx \frac{2T_{\rm c}t_{\rm w}}{1-\kappa} \tag{2.21}$$

where the last approximation is based on (2.20) and is valid for large $t_{\rm w}$. For nonzero $\mathbf{q}$, on the other hand,

$$C_{\mathbf{q}}(t_{\rm w},t_{\rm w}) = \frac{1}{g(t_{\rm w})}\left(\mathrm{e}^{-2\omega t_{\rm w}} + 2T_{\rm c}\int_0^{t_{\rm w}} \mathrm{d}t'\,g(t')\,\mathrm{e}^{-2\omega(t_{\rm w}-t')}\right) \approx \frac{T_{\rm c}}{\omega} \tag{2.22}$$

which is as expected since all nonzero Fourier modes eventually equilibrate. The crossover between the two limits takes place when $\omega t_{\rm w} \sim 1$, or $q \sim t_{\rm w}^{-1/2}$; physically, this represents the growth of the time-dependent correlation length as $\sim t_{\rm w}^{1/2}$. We therefore introduce the scaling variable $w = \omega t_{\rm w}$:

$$\frac{C_{\mathbf{q}}(t_{\rm w},t_{\rm w})}{C_{\mathbf{0}}(t_{\rm w},t_{\rm w})} = \frac{\mathrm{e}^{-2w} + 2T_{\rm c}\,\omega^{-1}\int_0^w \mathrm{d}y\,g(y/\omega)\,\mathrm{e}^{-2(w-y)}}{1 + 2T_{\rm c}\,\omega^{-1}\int_0^w \mathrm{d}y\,g(y/\omega)}. \tag{2.23}$$

Now keep w constant and let $t_{\rm w} \to \infty$, i.e. $\omega \to 0$. Then, $g(y/\omega) \sim (y/\omega)^{-\kappa}$ and the second terms dominate in denumerator and nominator to give

$$\frac{C_{\mathbf{q}}(t_{\rm w},t_{\rm w})}{C_{\mathbf{0}}(t_{\rm w},t_{\rm w})} = (1-\kappa)\int_0^1 \mathrm{d}y\,y^{-\kappa}\,\mathrm{e}^{-2w(1-y)}. \tag{2.24}$$

Combining (2.24) with (2.21) then gives the desired long-time scaling form

$$C_{\mathbf{q}}(t_{\rm w},t_{\rm w}) = \frac{T_{\rm c}}{\omega}\omega\mathcal{F}_C(\omega t_{\rm w}), \qquad \mathcal{F}_C(w) = 2w\int_0^1 \mathrm{d}y\,y^{-\kappa}\,\mathrm{e}^{-2w(1-y)}. \tag{2.25}$$

For $d > 4$ ($\kappa = 0$), this simplifies to $\mathcal{F}_C(w) = 1 - e^{-2w}$. As the derivation shows, equations (2.24) and (2.25) are valid whenever $t_w \gg 1$, even for $\omega = \mathcal{O}(1)$. The latter case corresponds to $w \to \infty$ and gives $\mathcal{F}_C(w) = 1$, which is indeed consistent with (2.22).

For quantities such as $C_{\mathbf{q}}(t_w, t_w)$ that depend only on a single time variable, what is meant by the long-time limit is unambiguous. For two-time quantities like $C_{\mathbf{q}}(t, t_w)$ we use the following terminology: the *long-time* limit refers to the regime $t \gg 1$ and $t_w \gg 1$ but without any restriction on $t - t_w$, which in particular is allowed to be short, i.e. of $\mathcal{O}(1)$. The *ageing* regime indicates more specifically the limit $t_w \to \infty$ at fixed ratio $x = t/t_w > 1$, which implies that $t - t_w$ is also large, of $\mathcal{O}(t_w)$. Occasionally, we specialize further to the regime of *well-separated* times, which corresponds to $t \gg t_w \gg 1$, i.e. the asymptotic behaviour of the ageing limit for $x \gg 1$.

To illustrate the difference, consider which wavevectors dominate the integral $\int (\mathrm{d}q) C_{\mathbf{q}}(t, t_w)$. In the long-time limit at equal times $t = t_w$, the scaling $C_{\mathbf{q}}(t_w, t_w) \sim 1/\omega$ for $\omega \gg 1/t_w$ combined with $(\mathrm{d}q) \sim \mathrm{d}\omega\, \omega^{d/2-1}$ for small ω shows that the integral is divergent at the upper end of the frequency regime $\omega = \mathcal{O}(t_w^{-1})$ for all $d > 2$; in other words, it is always dominated by values of ω (and therefore q) of $\mathcal{O}(1)$. This remains true for two-time correlations, as long as $t - t_w = \mathcal{O}(1)$. In the ageing limit, however, we have $t - t_w = \mathcal{O}(t_w) \gg 1$ and the exponential factor from $R_{\mathbf{q}}$ in (2.13) then ensures that only values of $\omega < (t - t_w)^{-1} = \mathcal{O}\big(t_w^{-1}\big)$ have to be considered in the integral.

3. Fluctuation–dissipation relations for finite-range observables

In this section, we consider FD relations for observables that probe correlations over a length scale that can be much larger than the lattice spacing, but remains much smaller than the system size. The latter can then be taken to infinity independently, so that the $\mathcal{O}(N^{-1/2})$-fluctuations of the Lagrange multiplier z become irrelevant. We begin by briefly considering spin observables, and then discuss bond observables in some more detail.

3.1. Spin observables

Since all observables that are linear in the spins can be written as superpositions of the Fourier modes $S_{\mathbf{q}}$, the basic ingredient for understanding the FD behaviour is the FDR for the latter. Using (2.11), this follows after a couple of lines as ($C' \equiv \partial_{t_w} C$)

$$X_{\mathbf{q}}(t, t_w) = \frac{T R_{\mathbf{q}}(t, t_w)}{C'_{\mathbf{q}}(t, t_w)} = T \left[2T - \left(\omega + \frac{g'(t_w)}{2g(t_w)} \right) C_{\mathbf{q}}(t_w, t_w) \right]^{-1}. \qquad (3.1)$$

This is *independent* of the later time t, a feature that is commonly observed in simple non-equilibrium models [2].

The fluctuating magnetization is simply $S_{\mathbf{0}}/N$, so setting $\mathbf{q} = \mathbf{0}$ in (3.1) directly gives the FDR for the magnetization

$$X_{\mathbf{0}}(t, t_w) = T \left[2T - \frac{g'(t_w)}{2g^2(t_w)} \left(1 + 2T \int_0^{t_w} \mathrm{d}t'\, g(t') \right) \right]^{-1}. \qquad (3.2)$$

At criticality, this converges on an $\mathcal{O}(1)$ time scale to the limit FDR

$$X^\infty = \frac{T_c}{2T_c + (\kappa/2)[2T_c/(1-\kappa)]} = \frac{1-\kappa}{2-\kappa} = \begin{cases} 1/2 & (d > 4) \\ 1 - 2/d & (d < 4) \end{cases} \qquad (3.3)$$

which is identical to the value obtained from the local magnetization [9] as one would expect on general grounds. Without working out the susceptibility explicitly, it is clear from the

t-independence of $X_{\mathbf{0}}(t, t_w)$ and its fast convergence to X^∞ that the limiting FD plot is a straight line. Both of these observations are exactly as in the Ising model in $d = 1$ [10]. Simulations have shown that also in the $d = 2$ Ising case the local–global correspondence holds for spin observables; the limiting FD plot is numerically indistinguishable from a straight line, though renormalization group arguments suggest that it should deviate slightly [11, 15, 17].

We should clarify that the Gaussian theory above applies directly not to the FDR for $S_{\mathbf{0}}$ but to the one for $S_{\mathbf{q}}$ with $q \ll t_w^{-1/2}$ but $q \gg L^{-1}$, where $L = N^{1/d}$ is the linear system size. The corresponding physical observable is a 'block' magnetization, i.e. the average of the spins within a block of size $\ell \sim 1/q$ much larger than the time-dependent correlation length $\sim t_w^{1/2}$ but still small compared to the overall system size. For $\mathbf{q} = \mathbf{0}$, i.e. $\ell = L$, one would in principle need to account for the non-Gaussian fluctuations. However, it turns out that these are negligible as long as the system is not magnetized on average (see section 8), so that the above results remain correct even for the global magnetization itself.

More generally, the FDR for any finite-range spin observable can be expressed as a superposition of those for the Fourier modes; this can be seen by arguments paralleling those in the $d = 1$ Ising case [10]. As there, one can then show that the asymptotic FDR that is approached for well-separated times $t \gg t_w \gg 1$ is dominated by the contribution from $\mathbf{q} = \mathbf{0}$, and hence identical to X^∞ calculated above [11]. At equal times, on the other hand, equilibrated modes with $q = \mathcal{O}(1)$ dominate and give $X = 1$. The crossover between these two regimes takes place when $t - t_w = \mathcal{O}(t_w)$ and follows (by superposition) from the corresponding crossover at $q = \mathcal{O}\big(t_w^{-1}\big)$ in the Fourier mode FDRs. From (2.25) and (3.1), the latter can be expressed as

$$X_{\mathbf{q}}(t, t_w) = \mathcal{F}_X(\omega t_w), \qquad \mathcal{F}_X^{-1}(w) = 2 - (2w - \kappa) \int_0^1 \mathrm{d}y\, y^{-\kappa}\, \mathrm{e}^{-2w(1-y)} \tag{3.4}$$

in the long-time limit, providing the expected interpolation between $X = X^\infty = 1/[2 + \kappa/(1 - \kappa)]$ for $w \to 0$ and $X = 1$ for $w \to \infty$.

3.2. Bond observables

Next, we consider bond energy observables, $\frac{1}{2}(S_i - S_j)^2$, where i and j are n.n. sites. Since all variables are Gaussian, the connected correlations follow by Wick's theorem. For the correlation of bond energies, one gets

$$C_{ij,kl}(t, t_w) = 2\tfrac{1}{4}\langle [S_i(t) - S_j(t)][S_k(t_w) - S_l(t_w)]\rangle^2 = \tfrac{1}{2}[C_{ik} - C_{il} - C_{jk} + C_{jl}]^2 \tag{3.5}$$

where time arguments have been left implicit. For the local case $(i, j) = (k, l)$, this simplifies to

$$C_{ij,ij} = 2[1 - C_{ij}]^2 \tag{3.6}$$

which tends to a nonzero constant for $t = t_w \to \infty$ since C_{ij} then approaches its equilibrium value, which is < 1.

Next, we turn to the response function. In general, if one perturbs the Hamiltonian H by $-hB\delta(t - t_w)$, then the equation of motion for S_i acquires an extra term $h(\partial B/\partial S_i)\delta(t - t_w)$. So, the perturbation in S_i is

$$\delta S_i(t) = h \sum_j R_{ij}(t, t_w) \frac{\partial B}{\partial S_j}(t_w). \tag{3.7}$$

Thus, the perturbation of an observable A is

$$\delta A(t) = h \sum_{ij} \frac{\partial A}{\partial S_i}(t) R_{ij}(t, t_w) \frac{\partial B}{\partial S_j}(t_w) \tag{3.8}$$

giving the response function [13]

$$R_{AB}(t,t_w) = \sum_{ij} R_{ij}(t,t_w)\left\langle \frac{\partial A}{\partial S_i}(t)\frac{\partial B}{\partial S_j}(t_w)\right\rangle. \tag{3.9}$$

For $A = \frac{1}{2}(S_i - S_j)^2$, $B = \frac{1}{2}(S_k - S_l)^2$, this yields

$$R_{ij,kl}(t,t_w) = [R_{ik} - R_{il} - R_{jk} + R_{jl}][C_{ik} - C_{il} - C_{jk} + C_{jl}]. \tag{3.10}$$

We now analyse the scaling of correlation, response and the resulting FDR. In terms of $C_{\mathbf{q}}(t,t_w)$, the bond correlation (3.5) is

$$C_{ij,kl}(t,t_w) = \frac{1}{2}\left[\int (\mathrm{d}q) C_{\mathbf{q}} (\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_k)} - \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_l)} - \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_k)} + \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_l)})\right]^2. \tag{3.11}$$

We can take out a factor $\exp(\mathrm{i}\mathbf{q}\cdot\Delta\mathbf{r})$ from all the exponentials, where $\Delta\mathbf{r} = \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j) + \frac{1}{2}(\mathbf{r}_k - \mathbf{r}_l)$ is the distance vector between the bond midpoints. In the remaining exponentials, $\mathbf{q}$ is multiplied by vectors with lengths of order unity.

Now assume $t - t_w \gg 1$. As explained above, integrals of two-time quantities over $\mathbf{q}$ are then dominated by the small-q regime, $q^2 \approx \omega < (t-t_w)^{-1}$. We can therefore Taylor expand in $\mathbf{q}$ and get, using the equivalence of the d lattice directions,

$$C_{ij,kl} = \frac{1}{2}\left[d^{-1}(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{r}_k - \mathbf{r}_l)\int (\mathrm{d}q) C_{\mathbf{q}} q^2\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\Delta\mathbf{r}}\right]^2. \tag{3.12}$$

Similarly, one finds for the response

$$R_{ij,kl} = [d^{-1}(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{r}_k - \mathbf{r}_l)]^2 \int (\mathrm{d}q) R_{\mathbf{q}} q^2\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\Delta\mathbf{r}} \int (\mathrm{d}q) C_{\mathbf{q}} q^2\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\Delta\mathbf{r}}. \tag{3.13}$$

For the *local* bond–bond correlation and response one sets $\Delta\mathbf{r} = \mathbf{0}$ and has $(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{r}_i - \mathbf{r}_j) = 1$, which gives for the FDR

$$X^{\mathrm{loc}}_{\mathrm{bond}}(t,t_w) = \frac{\int (\mathrm{d}q) T_c\, R_{\mathbf{q}} q^2}{\int (\mathrm{d}q) C'_{\mathbf{q}} q^2} = \frac{\int (\mathrm{d}q) R_{\mathbf{q}} q^2}{\int (\mathrm{d}q) X^{-1}_{\mathbf{q}} R_{\mathbf{q}} q^2}. \tag{3.14}$$

So, $1/X^{\mathrm{loc}}_{\mathrm{bond}}(t,t_w)$ can be thought of as an average of $X^{-1}_{\mathbf{q}}(t,t_w)$ over $\mathbf{q}$, with the weight $R_{\mathbf{q}}(t,t_w)q^{-2}$. The factor $R_{\mathbf{q}}$ ensures that significant contributions come only from wavevectors $\mathbf{q}$ up to length $q \sim (t-t_w)^{-1/2}$, i.e. up to $\omega(t-t_w) \approx 1$. Thus, when $t - t_w \ll t_w$, the result is dominated by the regime $\omega t_w \gg 1$, where $X_{\mathbf{q}} = 1$. For $t - t_w \gg t_w$, meanwhile, one only gets contributions from $\omega t_w \ll 1$, where $X_{\mathbf{q}} = X^\infty$. So, the FDR (3.14) for local-bond observables is a scaling function interpolating between 1 and X^∞, with the same X^∞ as for the magnetization. Explicitly one has, using (2.6) and changing integration variable from $\mathbf{q}$ to $w = \omega t_w$,

$$X^{\mathrm{loc}}_{\mathrm{bond}}(t,t_w) = \frac{\int_0^\infty \mathrm{d}w\, w^{d/2}\,\mathrm{e}^{-(x-1)w}}{\int_0^\infty \mathrm{d}w\, w^{d/2}\,\mathrm{e}^{-(x-1)w}\mathcal{F}_X^{-1}(w)} \tag{3.15}$$

with $x = t/t_w$ and $\mathcal{F}_X$ the scaling form (3.4) of $X_{\mathbf{q}}$. To find the shape of the FD plot, recall that the equal-time value of the local-bond correlation (3.6) is a constant in the long-time limit. For $t - t_w \gg 1$, on the other hand, equation (3.12) shows that $C^{\mathrm{loc}}_{\mathrm{bond}}$ scales as

$$C^{\mathrm{loc}}_{\mathrm{bond}}(t,t_w) \sim \frac{g(t_w)}{g(t)}\left[\int \mathrm{d}\omega\,\omega^{d/2-1}\frac{T_c}{\omega}\mathcal{F}_C(\omega t_w)\,\mathrm{e}^{-\omega(t-t_w)}\omega\right]^2 \tag{3.16}$$

$$\sim \frac{g(t_w)}{g(t)} t_w^{-d}\left[\int \mathrm{d}w\, w^{d/2-1}\mathcal{F}_C(w)\,\mathrm{e}^{-w(t-t_w)/t_w}\right]^2. \tag{3.17}$$

Since $\mathcal{F}_C(w) \to 1$ for $w \to \infty$, the w-integral would be divergent without the exponential cut-off and scales as $[(t-t_w)/t_w]^{-d/2}$ for $t - t_w \ll t_w$, so that $C^{\text{loc}}_{\text{bond}}(t, t_w) \sim (t-t_w)^{-d}$ in this regime. The regime $t - t_w > t_w$ where $X^{\text{loc}}_{\text{bond}}(t, t_w) \neq 1$ is therefore compressed into the region where $C^{\text{loc}}_{\text{bond}}$ is of order t_w^{-d}, so that the long-time limit of the FD plot is a straight line with equilibrium slope. Qualitatively, one thus has the same behaviour as for local-bond observables in the Ising model [10].

Next, consider *long-range* bond observables, where we sum (ij) and (kl) over all bonds. The same proviso as above for the magnetization applies here, i.e. by applying the Gaussian theory we are effectively considering the bond energies averaged over a block that is large but has to remain nonetheless small compared to the system size. One can show that the resulting equal-time correlation again approaches a constant value for $t_w \to \infty$. (This follows because for large $\Delta\mathbf{r}$, one can use the small-q expansion (3.12) even for equal times. From $C_{\mathbf{q}}(t_w, t_w) \approx T_c/q^2$, one gets $C(\Delta\mathbf{r}; t_w, t_w) \sim |\Delta\mathbf{r}|^{2-d}$ for large $\Delta\mathbf{r}$ and so $\int (\mathrm{d}q) C_{\mathbf{q}} q^2\, \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\Delta\mathbf{r}} \sim \nabla^2 |\Delta\mathbf{r}|^{2-d} \sim |\Delta\mathbf{r}|^{-d}$. The square $|\Delta\mathbf{r}|^{-2d}$ then yields a convergent sum over $\Delta\mathbf{r}$.) So we focus directly on the regime $t - t_w \gg 1$, where the expansion (3.12) is again valid. Keeping the bond (ij) fixed, the scalar product $(\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_k - \mathbf{r}_l)$ means that only bonds (kl) parallel to (ij) contribute, so that the sum over (kl) becomes a sum over $\Delta\mathbf{r}$, running over all lattice vectors. (For non-parallel bonds, $\Delta\mathbf{r}$ could also assume other values not corresponding to lattice vectors.) The sum over (ij) then just gives an overall factor of Nd. Normalizing by N, the block bond correlation function is

$$C^{\text{block}}_{\text{bond}}(t, t_w) = \frac{1}{2d}\frac{1}{N}\sum_{\Delta\mathbf{r}}\left[\int (\mathrm{d}q) C_{\mathbf{q}} q^2\, \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\Delta\mathbf{r}}\right]^2 = \frac{1}{2d}\int (\mathrm{d}q) C^2_{\mathbf{q}} q^4 \tag{3.18}$$

and similar arguments give for the (normalized) response

$$R^{\text{block}}_{\text{bond}}(t, t_w) = \frac{1}{d}\int (\mathrm{d}q) R_{\mathbf{q}} C_{\mathbf{q}} q^4 \tag{3.19}$$

so that the FDR becomes

$$X^{\text{block}}_{\text{bond}}(t, t_w) = \frac{\int (\mathrm{d}q) T_c\, R_{\mathbf{q}} C_{\mathbf{q}} q^4}{\int (\mathrm{d}q) C'_{\mathbf{q}} C_{\mathbf{q}} q^4} = \frac{\int (\mathrm{d}q) R_{\mathbf{q}} C_{\mathbf{q}} q^4}{\int (\mathrm{d}q) X^{-1}_{\mathbf{q}} R_{\mathbf{q}} C_{\mathbf{q}} q^4}. \tag{3.20}$$

Again, this is the inverse of a weighted average of $X^{-1}_{\mathbf{q}}$, now with weight $R_{\mathbf{q}} C_{\mathbf{q}} q^4$. The same arguments as for (3.14) then show that $X^{\text{block}}(t, t_w)$ scales with $x = t/t_w$ and interpolates between $X = 1$ for $x = 1$ and $X = X^\infty$ for $x \to \infty$. The value of the correlator (3.18) decays from $\mathcal{O}(1)$ at $t = t_w$ to $\mathcal{O}(t_w^{-d/2})$ at the point $x - 1 \approx 1$ where ageing effects appear. While this is larger than for the local-bond observables, it still decreases to zero for $t_w \to \infty$, so that the limiting FD plot is again of pseudo-equilibrium form. This is different to the case of the Ising model, where the global bond observables give nontrivial limiting FD plots [10].

In more detail, the scaling of the block bond correlator (3.18) in the ageing regime $t - t_w \gg 1$ is

$$C^{\text{block}}_{\text{bond}}(t, t_w) \sim \frac{g(t_w)}{g(t)}\int \mathrm{d}\omega\, \omega^{d/2-1}\frac{T_c^2}{\omega^2}\mathcal{F}^2_C(\omega t_w)\, \mathrm{e}^{-2\omega(t-t_w)}\omega^2 \tag{3.21}$$

$$\sim \left(\frac{t}{t_w}\right)^{\kappa} t_w^{-d/2}\int \mathrm{d}w\, w^{d/2-1}\mathcal{F}^2_C(w)\, \mathrm{e}^{-2(x-1)w}. \tag{3.22}$$

The integral scales as $(x-1)^{-d/2}$ for $x \approx 1$, so $C^{\text{block}}_{\text{bond}}(t, t_w) \sim (t-t_w)^{-d/2}$ there. For $x \gg 1$, on the other hand, the integral becomes $\sim x^{-(d+4)/2}$, so $C^{\text{block}}_{\text{bond}}(t, t_w) \sim t_w^{-\kappa+2} t^{\kappa-(d+4)/2}$. Explicitly, in this $t \gg t_w$ regime, $C^{\text{block}}_{\text{bond}} \sim t_w^2 t^{-(d+4)/2}$ for $d > 4$ and $C^{\text{block}}_{\text{bond}} \sim t_w^{d/2} t^{-d}$ for $d < 4$. The response function scales in the same way as $\partial_{t_w} C^{\text{block}}_{\text{bond}}$, because X is everywhere of order unity.

3.3. Product observables

Instead of the bond observables $\frac{1}{2}(S_i - S_j)^2$, we could consider the spin products $A = S_i S_j$, $B = S_k S_l$. The correlations are then

$$C_{ij,kl}(t, t_w) = \langle S_i(t)S_j(t)S_k(t_w)S_l(t_w)\rangle - \langle S_i(t)S_j(t)\rangle\langle S_k(t_w)S_l(t_w)\rangle \tag{3.23}$$

$$= C_{ik}(t, t_w)C_{jl}(t, t_w) + C_{il}(t, t_w)C_{jk}(t, t_w). \tag{3.24}$$

The local equal-time correlation function $C_{ij,ij}(t_w, t_w)$ thus approaches $1 + C_{ij}^2$ for $t_w \to \infty$. The corresponding response function is

$$R_{ij,kl}(t, t_w) = R_{ik}C_{jl} + R_{il}C_{jk} + R_{jk}C_{il} + R_{jl}C_{ik}. \tag{3.25}$$

In the *local* case, one can replace all functions by local ones in the ageing regime—there are no cancellations leading to extra factors of q^2 as was the case for bond observables, compare, e.g., (3.11) and (3.12)—so that the FDR

$$X_{\mathrm{prod}}^{\mathrm{loc}}(t, t_w) = \frac{4T_c R_{ii}C_{ii}}{4C_{ii}'C_{ii}} = \frac{T_c R_{ii}}{C_{ii}'} \tag{3.26}$$

becomes identical to the one for the spin autocorrelation and response. In particular, one again gets the same X^∞.

For the global (block) case, we can write

$$C_{\mathrm{prod}}^{\mathrm{block}}(t, t_w) = \sum_{(ij),(kl)} C_{ij,kl}(t, t_w) \tag{3.27}$$

$$= \frac{1}{4}\sum_{ijkl} n_{ij}n_{kl}[C_{ik}(t, t_w)C_{jl}(t, t_w) + C_{il}(t, t_w)C_{jk}(t, t_w)] \tag{3.28}$$

$$= \frac{1}{2}\sum_{ijkl} n_{ij}n_{kl}C_{ik}(t, t_w)C_{jl}(t, t_w) = \frac{1}{2}\int(\mathrm{d}q)n_{\mathbf{q}}^2C_{\mathbf{q}}^2(t, t_w) \tag{3.29}$$

where $n_{ij} = 1$ if i and j are nearest neighbours and 0 otherwise, and $n_{\mathbf{q}}$ is its Fourier transform. For the response, one has similarly

$$R_{\mathrm{prod}}^{\mathrm{block}}(t, t_w) = \int(\mathrm{d}q)n_{\mathbf{q}}^2R_{\mathbf{q}}(t, t_w)C_{\mathbf{q}}(t, t_w). \tag{3.30}$$

In the ageing regime, where $t - t_w \gg 1$, the integrals are dominated by small q, where $n_{\mathbf{q}}$ can be approximated by the constant $n_{\mathbf{0}} = 2d$. This cancels from the FDR, which becomes

$$X_{\mathrm{prod}}^{\mathrm{block}}(t, t_w) = \frac{\int(\mathrm{d}q)T_c R_{\mathbf{q}}C_{\mathbf{q}}}{\int(\mathrm{d}q)C_{\mathbf{q}}'C_{\mathbf{q}}}. \tag{3.31}$$

This is the inverse of the average of $X_{\mathbf{q}}^{-1}$ with weight $R_{\mathbf{q}}C_{\mathbf{q}}$. Again, this is a scaling function of t/t_w interpolating between 1 and the same X^∞ as for spin observables.

The scaling of the block product correlation function (3.29) itself is a little more complicated than for the bond observables and depends on dimensionality. Focusing again on $t - t_w \gg 1$, one has $C_{\mathrm{prod}}^{\mathrm{block}}(t, t_w) \approx 2d^2\int(\mathrm{d}q)C_{\mathbf{q}}^2(t, t_w)$. The integral defines the function $CC(t, t_w)$ discussed in section 6 for $d > 4$ and section 7 for $d < 4$. In the former case, one has from (6.8)–(6.10) that $CC(t, t_w) = CC_{\mathrm{eq}}(t - t_w)\mathcal{F}_{CC}(t/t_w)$ where $CC_{\mathrm{eq}}(t - t_w) \sim (t - t_w)^{(4-d)/2}$ asymptotically; this equilibrium contribution governs the behaviour of $C_{\mathrm{prod}}^{\mathrm{block}}(t, t_w)$ for $t - t_w \ll t_w$. Where ageing effects appear ($t - t_w \sim t_w$), $C_{\mathrm{prod}}^{\mathrm{block}} \sim t_w^{(4-d)/2}$ and so one gets a limiting pseudo-equilibrium FD plot. In the regime of well-separated times $x \gg 1$, the scaling function $\mathcal{F}_{CC}(x)$ decays as x^{-2} so that

$C_{\rm prod}^{\rm block}(t, t_{\rm w}) \sim t_{\rm w}^2 t^{-d/2}$. These scalings, though not the overall magnitude of $C_{\rm prod}^{\rm block}$, are the same as for the energy correlation function C_E in (6.14): both functions are proportional to $CC(t, t_{\rm w})$ in the ageing regime.

In the opposite case $d < 4$, the equal-time value $CC(t, t)$ (and therefore $C_{\rm prod}^{\rm block}(t, t)$) diverges as $t^{(4-d)/2}$, see (7.4). The normalized correlator $CC(t, t_{\rm w})/CC(t, t)$ is a scaling function $\mathcal{G}(x)$ of $x = t/t_{\rm w}$, implying that the normalized FD plot will approach a nontrivial limit form, with asymptotic slope X^∞ as shown above. Quantitatively, because $\mathcal{G}(x) \sim x^{-d/2}$ for $x \gg 1$, one has $C_{\rm prod}^{\rm block}(t, t_{\rm w}) \sim t^{(4-d)/2}(t_{\rm w}/t)^{d/2} \sim t_{\rm w}^{d/2} t^{2-d}$ for $t \gg t_{\rm w}$.

4. Correlation and response for global observables

We now ask what happens if we go from block observables to truly global ones, which reflect properties averaged over the entire system; the total energy is an important example. We anticipate that here non-Gaussian fluctuations are important. Indeed, the results above show that this must be case. Otherwise we could directly extend the Gaussian theory results from block to global observables, with no change to correlation and response functions. The global *bond* observable is just the energy. Using the spherical constraint, this can be written as

$$E = \sum_{(ij)} \frac{1}{2}(S_i - S_j)^2 = N - \sum_{(ij)} S_i S_j \tag{4.1}$$

and so is identical to the global *spin product* observable, up to a trivial additive constant and sign. So the global bond and product observables must have identical correlation and response functions; but we noted above that this requirement is not satisfied by the Gaussian theory. Thus, non-Gaussian fluctuations are essential to get correct results for global observables.

Physically, the origin of the distinction between block observables and global ones is the effective infinite-range interaction induced by the spherical constraint. In a model with short-range interactions, block observables will show identical behaviour to global ones whenever the block size ℓ is larger than any correlation length in the system, whether or not $\ell \ll L$: the behaviour of any large subsystem is equivalent to that of the system as a whole. In the spherical model, the infinite-range interaction breaks this connection, and global correlation and response functions cannot be deduced from those for block observables.

4.1. Non-Gaussian fluctuations

To make progress, we need to return to the original equation of motion (2.2). This can be written as

$$\partial_t S_i = -\sum_j \Omega_{ij} S_j + \xi_i - (z(t) + N^{-1/2}\Delta z) S_i \tag{4.2}$$

where the notation emphasizes that the fluctuating contribution to the Lagrange multiplier is of $\mathcal{O}(N^{-1/2})$. The latter induces non-Gaussian fluctuations in S_i of the same order. This quantitatively shows why the Gaussian theory works for block observables: as long as one considers correlations of a number of spins that is $\ll N$, fluctuations of $\mathcal{O}(N^{-1/2})$ can be neglected. For global observables, on the other hand, we require the correlations of all N spins and the Gaussian approximation then becomes invalid.

To account systematically for non-Gaussian effects, we represent the spins as $S_i = s_i + N^{-1/2} r_i$, where s_i gives the limiting result for $N \to \infty$, which has purely Gaussian statistics, and $N^{-1/2} r_i$ is a leading-order fluctuation correction which will be non-Gaussian. Inserting this decomposition into (4.2) and collecting terms of $\mathcal{O}(1)$ and $\mathcal{O}(N^{-1/2})$ gives

$\partial_t s_i = -\Omega_{ij}s_j - z(t)s_i + \xi_i$ as expected; to lighten the notation we use the summation convention for repeated indices from now on. For the non-Gaussian corrections, one gets the equation of motion

$$\partial_t r_i = -\Omega_{ij} r_j - z(t) r_i - \Delta z s_i \tag{4.3}$$

with solution

$$r_i(t) = R_{ij}(t,0) r_j(0) - \int_0^t \mathrm{d}t'\, R_{ij}(t,t') s_j(t') \Delta z(t'). \tag{4.4}$$

The properties of $\Delta z(t')$ can now be determined from the requirement that, due to the spherical constraint, $N^{-1}\sum_i S_i^2(t) = 1$ at all times. Inserting $S_i = s_i + N^{-1/2} r_i$ and expanding to the leading order in $N^{-1/2}$ gives the condition

$$\frac{1}{N}\sum_i s_i(t) r_i(t) = -\frac{1}{2} N^{-1/2} \sum_i \left(s_i^2(t) - 1\right) \equiv -\frac{1}{2}\Delta(t) \tag{4.5}$$

where the last equality defines $\Delta(t)$, a fluctuating quantity of $\mathcal{O}(1)$ that describes the (normalized) fluctuations of the squared length of the Gaussian spin vector s_i. At $t = 0$, condition (4.5) is solved to leading order by setting $r_i(0) = -\frac{1}{2}\Delta(0) s_i(0)$, since $(1/N)\sum_i s_i^2(0) = 1 + \mathcal{O}(N^{-1/2})$. With this assignment, and setting $a(t) = 2\Delta z(t) + \Delta(0)\delta(t)$, equation (4.4) reads

$$r_i(t) = -\frac{1}{2}\int \mathrm{d}t'\, R_{ij}(t,t') s_j(t') a(t'). \tag{4.6}$$

We have left the integral limits unspecified here: the factor R_{ij} automatically enforces $t' < t$, and we use the convention $a(t') = 0$ for $t' < 0$. The spherical constraint condition (4.5) then becomes

$$\int \mathrm{d}t'\, \frac{1}{N} s_i(t) R_{ij}(t,t') s_j(t') a(t') = \Delta(t). \tag{4.7}$$

Now, up to fluctuations of $\mathcal{O}(N^{-1/2})$ which are negligible to leading order (even if they are correlated with $a(t')$), we can replace $(1/N) s_i(t) R_{ij}(t,t') s_j(t')$ by its average

$$K(t,t') \equiv \frac{1}{N}\langle s_i(t) R_{ij}(t,t') s_j(t')\rangle = \frac{1}{N} R_{ij}(t,t') C_{ij}(t,t') = \int (\mathrm{d}q) R_{\mathbf{q}}(t,t') C_{\mathbf{q}}(t,t'). \tag{4.8}$$

If we then define the inverse operator, L, of K via

$$\int \mathrm{d}t'\, K(t,t') L(t', t_{\mathrm{w}}) = \delta(t - t_{\mathrm{w}}) \tag{4.9}$$

for $t_{\mathrm{w}} \geqslant 0$, then the solution to (4.7) is

$$a(t) = \int \mathrm{d}t'\, L(t,t') \Delta(t') \tag{4.10}$$

where for consistency we adopt the convention $\Delta(t') = 0$ for $t' < 0$. With (4.6) we then get an explicit expression for the non-Gaussian $\mathcal{O}(N^{-1/2})$-corrections to the spins,

$$r_i(t) = -\frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t''\, R_{ij}(t,t') s_j(t') L(t',t'') \Delta(t''), \tag{4.11}$$

in terms of the properties of the uncorrected Gaussian spins s_i.

4.2. The functions K and L

Before proceeding, we analyse the properties of K and L. From (4.8), $K(t,t')$ vanishes for $t < t'$ while its limit value for $t \to t'^{+}$ is $(1/N)\delta_{ij}C_{ij}(t,t) = (1/N)C_{ii}(t,t) = 1$. Inserting (2.6), (2.11) and (2.13) into (4.8), one also finds that the equal-time slope has the simple value $\partial_{t'}K(t,t')|_{t=t'^{+}} = 2T$. From these properties and definition (4.9), it follows that

$$L(t,t') = \delta'(t-t') + L^{(1)}(t,t'), \qquad L^{(1)}(t,t') = 2T\delta(t-t') - L^{(2)}(t,t') \tag{4.12}$$

where $L^{(2)}(t,t')$ vanishes for $t < t'$ and jumps to a finite value at $t = t'^{+}$; otherwise it is smooth and, as we will later see, positive. The structure of (4.12) can be easily verified, e.g., for the limit of equilibrium at high temperature T, where $z_{\mathrm{eq}} = T$ and all ω can be neglected compared to z_{eq}. One then has $K(t,t') = \exp(-2T(t-t'))$ and the inverse (4.9) can be calculated by Laplace transform. Since the Laplace transform of $K(t,t')$ is $\hat{K}(s) = 1/(s+2T)$ this gives $\hat{L}(s) = s + 2T$, which corresponds to (4.12) with $L^{(2)} \equiv 0$.

We next determine the long-time forms of K and $L^{(2)}$ for quenches to criticality. In both cases, it is useful to factor out the equilibrium contribution. For K this is, from (4.8) and using (2.6) and (2.13),

$$K_{\mathrm{eq}}(t-t_{\mathrm{w}}) = \int (\mathrm{d}q)\, \mathrm{e}^{-2\omega(t-t_{\mathrm{w}})}\frac{T_{\mathrm{c}}}{\omega}. \tag{4.13}$$

Apart from the factor of 2 in the time argument, this is just the (critical) equilibrium spin–spin autocorrelation function. One can also write $K(t) = 2T_{\mathrm{c}}\int_t^{\infty}\mathrm{d}t'\, f(t')$ from (2.15) and this shows that $K_{\mathrm{eq}}(t) \sim t^{(2-d)/2}$ for large time differences. The ratio $K(t,t_{\mathrm{w}})/K_{\mathrm{eq}}(t-t_{\mathrm{w}})$ will show deviations from 1 when ageing effects appear, i.e. when $t - t_{\mathrm{w}} \sim t_{\mathrm{w}}$. The form of these deviations can be worked out by using the scaling form (2.25) of $C_{\mathbf{q}}(t_{\mathrm{w}},t_{\mathrm{w}})$ and recalling that only the small-q regime contributes, where $(\mathrm{d}q) \sim \mathrm{d}\omega\,\omega^{d/2-1}$. Changing integration variable to $w = \omega t_{\mathrm{w}}$ gives

$$K(t,t_{\mathrm{w}}) = K_{\mathrm{eq}}(t-t_{\mathrm{w}})\mathcal{F}_K(t/t_{\mathrm{w}}) \tag{4.14}$$

$$\mathcal{F}_K(x) = x^{\kappa}\frac{\int \mathrm{d}w\, w^{(d-4)/2}\,\mathrm{e}^{-2(x-1)w}\mathcal{F}_C(w)}{\int \mathrm{d}w\, w^{(d-4)/2}\,\mathrm{e}^{-2(x-1)w}} \tag{4.15}$$

where the first factor in (4.15) arises from the two factors of $[g(t_{\mathrm{w}})/g(t)]^{1/2}$ contributed by $R_{\mathbf{q}}(t,t_{\mathrm{w}})$ and $C_{\mathbf{q}}(t,t_{\mathrm{w}})$, respectively. By construction, $\mathcal{F}_K(x)$ should approach 1 for $x \to 1$; indeed, in this limit the w-integrals are dominated by large values of $w \sim 1/(x-1)$, for which $\mathcal{F}_C = 1$. The decay for large x follows from $\mathcal{F}_C(w) \sim w$ for small w as $\mathcal{F}_K(x) \sim x^{\kappa-1}$. Explicitly, one finds by using (2.25) and carrying out the w-integrals that

$$\mathcal{F}_K(x) = \frac{d-2}{2}(x-1)^{(d-2)/2}x^{\kappa}\int_0^1 \mathrm{d}y\, y^{-\kappa}(x-y)^{-d/2}. \tag{4.16}$$

For $d > 4$, where $\kappa = 0$, this gives

$$\mathcal{F}_K(x) = 1 - \left(\frac{x-1}{x}\right)^{(d-2)/2} \qquad (d > 4) \tag{4.17}$$

while for $d < 4$ the required indefinite integral is $[(d-2)/2]\int \mathrm{d}y\, y^{(d-4)/2}(x-y)^{-d/2} = -x^{-1}(x/y-1)^{(2-d)/2}$ and one gets simply

$$\mathcal{F}_K(x) = x^{(2-d)/2} \qquad (d < 4). \tag{4.18}$$

Next, we determine $L^{(2)}$. Combining (4.9) and (4.12), the defining equation is

$$\int \mathrm{d}t'\, K(t,t')L^{(2)}(t',t_{\mathrm{w}}) = 2TK(t,t_{\mathrm{w}}) - \partial_{t_{\mathrm{w}}}K(t,t_{\mathrm{w}}). \tag{4.19}$$

Again it makes sense to extract the equilibrium part of $L^{(2)}$. This is defined by

$$\int \mathrm{d}t'\, K_{\mathrm{eq}}(t-t')L^{(2)}_{\mathrm{eq}}(t'-t_{\mathrm{w}}) = 2TK_{\mathrm{eq}}(\Delta t) + K'_{\mathrm{eq}}(\Delta t) \tag{4.20}$$

where $\Delta t = t - t_{\mathrm{w}}$. Solving by Laplace transform gives

$$\hat{L}^{(2)}_{\mathrm{eq}}(s) = \frac{2T\hat{K}_{\mathrm{eq}}(s) + (s\hat{K}_{\mathrm{eq}}(s) - 1)}{\hat{K}_{\mathrm{eq}}(s)} = s + 2T - \frac{1}{\hat{K}_{\mathrm{eq}}(s)} \tag{4.21}$$

where from (4.13), at criticality,

$$\hat{K}_{\mathrm{eq}}(s) = T_{\mathrm{c}} \int (\mathrm{d}q) \frac{1}{\omega(s+2\omega)}. \tag{4.22}$$

The leading small-s behaviour of this is $\hat{K}_{\mathrm{eq}}(0) - \hat{K}_{\mathrm{eq}}(s) \sim s^{(d-4)/2}$ for $d > 4$ (plus, for $d > 6$, additional analytic terms of integer order in s which are irrelevant for us). For $d < 4$, on the other hand, $\hat{K}_{\mathrm{eq}}(s) \sim s^{(d-4)/2}$ is divergent for $s \to 0$. Inserting these scalings into (4.21) and inverting the Laplace transform gives for the asymptotic behaviour of $L^{(2)}_{\mathrm{eq}}$

$$L^{(2)}_{\mathrm{eq}}(t) \sim \begin{cases} t^{(2-d)/2} & (d > 4) \\ t^{(d-6)/2} & (d < 4). \end{cases} \tag{4.23}$$

It will be important below that, for $d > 4$, $K_{\mathrm{eq}}(t)$ and $L^{(2)}_{\mathrm{eq}}(t)$ both decay asymptotically as $t^{(2-d)/2}$. The ratio between them can be worked out from (4.21), by expanding for small s as $1/\hat{K}_{\mathrm{eq}}(s) \approx 1/[\hat{K}_{\mathrm{eq}}(0) - cs^{(d-4)/2}] = 1/\hat{K}_{\mathrm{eq}}(0) + cs^{(d-4)/2}/\hat{K}^2_{\mathrm{eq}}(0)$ where c is some constant; comparing with $\hat{K}_{\mathrm{eq}}(s) \approx \hat{K}_{\mathrm{eq}}(0) - cs^{(d-4)/2}$ gives

$$L^{(2)}_{\mathrm{eq}}(t) = K_{\mathrm{eq}}(t)/\hat{K}^2_{\mathrm{eq}}(0) \tag{4.24}$$

for large time differences t.

The integral of $L^{(2)}_{\mathrm{eq}}(t)$ over all times follows from (4.21) as

$$\hat{L}^{(2)}_{\mathrm{eq}}(0) = \int_0^\infty \mathrm{d}t\, L^{(2)}_{\mathrm{eq}}(t) = \begin{cases} 2T_{\mathrm{c}} - \hat{K}^{-1}_{\mathrm{eq}}(0) = 2T_{\mathrm{c}}[1 - 1/\int(\mathrm{d}q)(T_{\mathrm{c}}/\omega)^2] & (d > 4) \\ 2T_{\mathrm{c}} & (d < 4). \end{cases} \tag{4.25}$$

Using the fact that $\int(\mathrm{d}q)(T_{\mathrm{c}}/\omega) = 1$, one has $\int(\mathrm{d}q)(T_{\mathrm{c}}/\omega)^2 > 1$ so that $\hat{L}^{(2)}_{\mathrm{eq}}(0)$ is positive independently of d. This is consistent with the intuition that, with the sign as chosen in (4.12), the function $L^{(2)}$ is positive.

With the equilibrium part of $L^{(2)}$ determined we make a long-time scaling ansatz for $L^{(2)}$,

$$L^{(2)}(t, t_{\mathrm{w}}) = L^{(2)}_{\mathrm{eq}}(t - t_{\mathrm{w}})\mathcal{F}_L(t/t_{\mathrm{w}}), \tag{4.26}$$

so that (4.19) becomes

$$\int \mathrm{d}t'\, K_{\mathrm{eq}}(t-t')L^{(2)}_{\mathrm{eq}}(t'-t_{\mathrm{w}})\mathcal{F}_K(t/t')\mathcal{F}_L(t'/t_{\mathrm{w}}) = 2T_{\mathrm{c}}K_{\mathrm{eq}}(\Delta t)\mathcal{F}_K(x) + K'_{\mathrm{eq}}(\Delta t)\mathcal{F}_K(x) + \frac{t}{t_{\mathrm{w}}^2}K_{\mathrm{eq}}(\Delta t)\mathcal{F}'_K(x) \tag{4.27}$$

where $\Delta t = t - t_{\mathrm{w}}$ and $x = t/t_{\mathrm{w}}$ as before. We now take the ageing limit of large t_{w} with $\Delta t = \mathcal{O}(t_{\mathrm{w}})$ to determine $\mathcal{F}_L$. The second and third terms on the rhs are then smaller by factors of order $1/t_{\mathrm{w}}$ than the first and can be neglected to leading order. The second term on the rhs of (4.20) is likewise subdominant, and this can be used to rewrite the dominant first term on the rhs of (4.27), giving

$$\mathcal{F}_K(x) = \frac{\int \mathrm{d}t'\, K_{\mathrm{eq}}(t-t')L^{(2)}_{\mathrm{eq}}(t'-t_{\mathrm{w}})\mathcal{F}_K(t/t')\mathcal{F}_L(t'/t_{\mathrm{w}})}{\int \mathrm{d}t'\, K_{\mathrm{eq}}(t-t')L^{(2)}_{\mathrm{eq}}(t'-t_{\mathrm{w}})}. \tag{4.28}$$

We consider first $d > 4$. Then, both the functions $K_{\rm eq}(\Delta t)$ and $L^{(2)}_{\rm eq}(\Delta t)$ have finite integrals $\hat{K}_{\rm eq}(0)$ and $\hat{L}^{(2)}_{\rm eq}(0)$, respectively, over $\Delta t = 0 \ldots \infty$. In the ageing limit, the factors $K_{\rm eq}(t-t')$ and $L^{(2)}_{\rm eq}(t'-t_{\rm w})$ therefore act to concentrate the mass of the integrals appearing in (4.28) around $t' = t_{\rm w}$ and $t' = t$. This can be seen more formally by changing to $y = t'/t_{\rm w}$ as the integration variable and taking $t_{\rm w}$ large. Then, the factors $K_{\rm eq}(t_{\rm w}(x-y))$ and $L^{(2)}_{\rm eq}(t_{\rm w}(y-1))$ produce singularities $\sim(x-y)^{(2-d)/2}$ for $y \to x$ and $\sim(y-1)^{(2-d)/2}$ for $y \to 1$, respectively, and because these are non-integrable they dominate the integral for $t_{\rm w} \to \infty$. All other factors in the integrals are slowly varying near the relevant endpoints and can be replaced by their values there. In the ageing limit, we can therefore write (4.28) as

$$\mathcal{F}_K(x) = \frac{\hat{K}_{\rm eq}(0)L^{(2)}_{\rm eq}(\Delta t)\mathcal{F}_K(1)\mathcal{F}_L(x) + K_{\rm eq}(\Delta t)\hat{L}^{(2)}_{\rm eq}(0)\mathcal{F}_K(x)\mathcal{F}_L(1)}{\hat{K}_{\rm eq}(0)L^{(2)}_{\rm eq}(\Delta t) + K_{\rm eq}(\Delta t)\hat{L}^{(2)}_{\rm eq}(0)}. \tag{4.29}$$

Equation (4.24) tells us that the Δt-dependent factors cancel, giving together with (4.25) and $\mathcal{F}_K(1) = \mathcal{F}_L(1) = 1$

$$\mathcal{F}_K(x) = \frac{\hat{K}^{-1}_{\rm eq}(0)\mathcal{F}_L(x) + \hat{L}^{(2)}_{\rm eq}(0)\mathcal{F}_K(x)}{\hat{K}^{-1}_{\rm eq}(0) + \hat{L}^{(2)}_{\rm eq}(0)} = \frac{\mathcal{F}_L(x) + [2T_{\rm c}\hat{K}_{\rm eq}(0) - 1]\mathcal{F}_K(x)}{2T_{\rm c}\hat{K}_{\rm eq}(0)}. \tag{4.30}$$

In $d > 4$, where $\hat{K}_{\rm eq}(0)$ is finite, we therefore have the simple result that the scaling functions for K and $L^{(2)}$ are identical,

$$\mathcal{F}_L(x) = \mathcal{F}_K(x). \tag{4.31}$$

But in the limit $d \to 4$ from above, $\hat{K}_{\rm eq}(0)$ diverges and (4.30) gives no information about $\mathcal{F}_L$. For $d < 4$, a different approach is therefore needed to determine $\mathcal{F}_L$. One subtracts from (4.27) the first and second terms on its rhs, using (4.20) to rewrite them as an integral and changing integration variable from t' to $y = t'/t_{\rm w}$. This gives

$$t_{\rm w}\int_1^x {\rm d}y\, K_{\rm eq}(t_{\rm w}(x-y))L^{(2)}_{\rm eq}(t_{\rm w}(y-1))[\mathcal{F}_K(x/y)\mathcal{F}_L(y) - \mathcal{F}_K(x)] = \frac{t}{t_{\rm w}^2}K_{\rm eq}(\Delta t)\mathcal{F}'_K(x). \tag{4.32}$$

For $y \to x$, $K_{\rm eq}(t_{\rm w}(x-y))$ contributes a singularity $\sim(x-y)^{(2-d)/2}$ which is integrable in $d < 4$. For $y \to 1$, the terms in square brackets vanish as $\sim y-1$ since $\mathcal{F}_L(y)$ is smooth at $y = 1$ as we will see below, in the sense that $\mathcal{F}'_L(1)$ is finite. These terms combine with the $\sim(y-1)^{(d-6)/2}$ from $L^{(2)}_{\rm eq}$ to give an integrable $\sim(y-1)^{(d-4)/2}$. The contributions from the short-time behaviour of $K_{\rm eq}$ and $L^{(2)}_{\rm eq}$ are therefore unimportant in the ageing limit and we can replace these functions by their power-law asymptotes. Up to overall d-dependent numerical factors condition (4.32) then becomes

$$t_{\rm w}^{-1}\int_1^x {\rm d}y(x-y)^{(2-d)/2}(y-1)^{(d-6)/2}[\mathcal{F}_K(x/y)\mathcal{F}_L(y) - \mathcal{F}_K(x)] = \frac{t}{t_{\rm w}^2}\Delta t^{(2-d)/2}\mathcal{F}'_K(x). \tag{4.33}$$

In the ageing limit Δt scales as $t_{\rm w}$, and so the lhs of this equation $\left(\sim t_{\rm w}^{-1}\right)$ becomes large compared to the rhs $\left(\sim t_{\rm w}^{-d/2}\right)$ unless the y-integral vanishes. The required condition for $\mathcal{F}_L$ is therefore

$$\int_1^x {\rm d}y(x-y)^{(2-d)/2}(y-1)^{(d-6)/2}[\mathcal{F}_K(x/y)\mathcal{F}_L(y) - \mathcal{F}_K(x)] = 0. \tag{4.34}$$

This is in principle an integral equation for $\mathcal{F}_L$. Fortunately, however, the solution is the naive extension of (4.31) to $d < 4$: with $\mathcal{F}_K(x) = x^{(2-d)/2}$ from (4.18) one sees that for

$\mathcal{F}_L(x) = \mathcal{F}_K(x) = x^{(2-d)/2}$ the square bracket in (4.34) vanishes identically. The identity (4.31) therefore holds both for $d < 4$ and for $d > 4$.

In summary, we have determined long-time scaling forms for K and L for quenches to criticality. For K, the result is (4.14) with (4.13) and (4.17), (4.18); for L, we have (4.26) with (4.21), (4.23) and (4.31). Combined with (4.11), this fully determines the leading non-Gaussian corrections to the spherical model dynamics (at long times, and after a quench to criticality from a random initial state).

5. General expressions for energy correlation and response

In this section, we derive general expressions for the two-time correlation and response functions of the energy, taking into account non-Gaussian fluctuations. The results will be valid for arbitrary quenches since we will leave K and L unspecified.

5.1. Energy correlation function

We can write the energy (2.1) as $H = \frac{1}{2} S_i \Omega_{ij} S_j$. Inserting $S_i = s_i + N^{-1/2} r_i$, the energy correlation function (normalized by N) is to leading order

$$C_E(t, t_w) = \frac{1}{4N} \langle (s_i \Omega_{ij} s_j + 2N^{-1/2} s_i \Omega_{ij} r_j)|_t (s_k \Omega_{kl} s_l + 2N^{-1/2} s_k \Omega_{kl} r_l)|_{t_w} \rangle'. \tag{5.1}$$

Using (4.11), all quantities involved can be expressed in terms of the Gaussian variables s_i so that the average can be performed using Wick's theorem, i.e. by taking products of all possible pairings. We use the prime on the average to indicate the connected correlation function. This just means that in the Wick expansion all terms not containing any pairings of a variable at t with one at t_w have to be discarded, since these terms give the disconnected contribution $\langle (\cdots)|_t \rangle \langle (\cdots)|_{t_w} \rangle$. Multiplying out (5.1) one obtains four contributions. The first one is

$$4C_E^{(1)} = \frac{1}{N} \Omega_{ij} \Omega_{kl} \langle s_i(t) s_j(t) s_k(t_w) s_l(t_w) \rangle' = \frac{2}{N} \Omega_{ij} \Omega_{kl} C_{jk}(t, t_w) C_{il}(t, t_w). \tag{5.2}$$

To eliminate one of the factors of Ω, note from (2.6) that $(\partial_t + z(t)) R_{\mathbf{q}}(t, t_w) = -\omega R_{\mathbf{q}}(t, t_w)$ for $t > t_w$. In real space, this reads

$$(\partial_t + z(t)) R_{ik}(t, t_w) = -\Omega_{ij} R_{jk}(t, t_w) = -R_{ij}(t, t_w) \Omega_{jk} \tag{5.3}$$

and from (2.13) an exactly analogous relation holds for $C_{ij}(t, t_w)$. Thus,

$$4C_E^{(1)} = -\frac{2}{N} [(\partial_t + z(t)) C_{ik}(t, t_w)] \Omega_{kl} C_{il}(t, t_w) \tag{5.4}$$

$$= -\frac{1}{N} (\partial_t + 2z(t)) C_{ik}(t, t_w) \Omega_{kl} C_{il}(t, t_w) \equiv -(\partial_t + 2z(t)) C\Omega C(t, t_w). \tag{5.5}$$

The last equality defines $C\Omega C$, which is just the normalized trace of the product of the matrices $C_{ik}(t, t_w)$, Ω_{kl} and $C_{il} = C_{li}$; in Fourier space, $C\Omega C(t, t_w) = \int (\mathrm{d}q) \omega C_{\mathbf{q}}^2(t, t_w)$.

The second contribution to C_E is

$$4C_E^{(2)} = \frac{2}{N^{3/2}} \Omega_{ij} \Omega_{kl} \langle s_i(t) r_j(t) s_k(t_w) s_l(t_w) \rangle' \tag{5.6}$$

$$= -\frac{1}{N} \int \mathrm{d}t' \, \mathrm{d}t'' \Omega_{ij} \Omega_{kl} R_{jm}(t, t') L(t', t'') \langle s_i(t) s_m(t') N^{-1/2} \Delta(t'') s_k(t_w) s_l(t_w) \rangle' \tag{5.7}$$

where we have inserted (4.11). Writing $N^{-1/2}\Delta(t'')$ explicitly this becomes

$$4C_E^{(2)} = -\frac{1}{N^2} \int \mathrm{d}t' \, \mathrm{d}t'' \Omega_{ij} \Omega_{kl} R_{jm}(t, t') L(t', t'') \big\langle s_i(t) s_m(t') \big[s_n^2(t'') - \big\langle s_n^2(t'') \big\rangle \big] s_k(t_w) s_l(t_w) \big\rangle'. \tag{5.8}$$

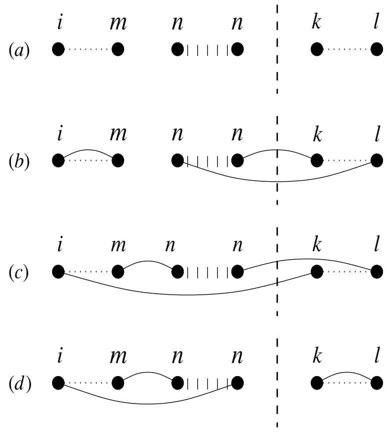

Figure 1. Illustration of Wick pairings for $C_E^{(2)}$. (*a*) The six Gaussian spins that need to be paired in (5.8) are indicated by circles with site labels. Spins arising from the expansion of $H(t)$ and $H(t_\mathrm{w})$ are to the left and right of the vertical dashed line, respectively; any pairing that contributes to the connected correlator must have links across this line. Dotted lines connect spins that are already coupled in space: s_i and s_m via the factor $\Omega_{ij}R_{jm}$, and s_k and s_l via Ω_{kl}. The vertical lines between s_n and s_n indicate that pairings which couple these spins are not allowed. (*b*) The solid lines show the only Wick pairing that contributes to leading order in $1/N$: it gives two independent groups of spins. Part (*c*) only gives one group and so is subleading. Part (*d*) has two groups but is excluded from the connected correlator because there are no pairs across the dashed line.

We now need to perform the Wick expansion of the average. The subtraction $s_n^2 - \langle s_n^2 \rangle$ means that all terms which would pair s_n with s_n are excluded; s_k and s_l also cannot be paired because we are considering the connected correlation. We can reduce the number of pairings further by bearing in mind that we need to get an overall result of $\mathcal{O}(1)$. The index j does not need to be considered further: after summing over j, $\Omega_{ij}R_{jm}$ is some translationally invariant function of the distance vector between spins s_i and s_m. If the remaining indices i, k, l, m, n where unrestricted, then together with the $1/N^2$ prefactor we would maximally get an $\mathcal{O}(N^3)$ result. Each of the factors $\Omega_{ij}R_{jm}$ and Ω_{kl} couples two indices and so reduces the order of the result by $1/N$. Having already got two such couplings outside the average, we can only 'afford' one extra coupling from the Wick pairings to get a contribution of $\mathcal{O}(1)$. After some reflection one sees that this only leaves the pairing $[im][kn][ln]$: $[im]$ introduces no further coupling beyond $\Omega_{ij}R_{jm}$, and $[kn][ln]$ gives only one further coupling beyond Ω_{kl}. Alternatively, we can think of this pairing as having the indices i, m and k, l, n, n in two independent groups; each group gives a factor of N and this just cancels the $1/N^2$ prefactor. All other allowed Wick pairings give smaller terms, as illustrated graphically in figures 1(*a*) and (*b*). For example, $[ik][mn][ln]$ together with $\Omega_{ij}R_{jm}$ and Ω_{kl} couples *all* indices into a single group and thus gives a term of only $\mathcal{O}(1/N)$ (figure 1(*c*)). The pairing $[in][mn][kl]$ would give two independent groups and thus an $\mathcal{O}(1)$ term, but is excluded because k and l cannot be paired in the connected correlator (figure 1(*d*)). Bearing in mind that our dominant pairing has a symmetry factor of 2 because the s_n's in $[kn][ln]$ can be swapped, we have thus finally

$$4C_E^{(2)} = -2\int \mathrm{d}t'\,\mathrm{d}t''\frac{1}{N}\Omega_{ij}R_{jm}(t,t')C_{im}(t,t')L(t',t'')\frac{1}{N}\Omega_{kl}C_{nk}(t'',t_\mathrm{w})C_{nl}(t'',t_\mathrm{w}) \tag{5.9}$$

$$= \int \mathrm{d}t'\,\mathrm{d}t''[(\partial_t + 2z(t))K(t,t') - \delta(t-t')]L(t',t'')C\Omega C(t'',t_\mathrm{w}). \tag{5.10}$$

In going to the last line we have exploited (5.3) to eliminate Ω_{ij}. Since t' is an integration variable, we have also been careful here to subtract off with $\delta(t-t')$ the spurious contribution

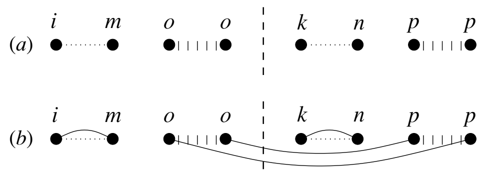

Figure 2. Wick pairings for $C_E^{(4)}$. (*a*) The constraints for the possible pairings in (5.16). (*b*) The only pairing that contributes to leading order, forming three independent groups of spins.

which ∂_t applied to the step discontinuity in $K(t, t')$ would otherwise give. The t'-integration can now be carried out using (4.9) and we get

$$4C_E^{(2)} = (\partial_t + 2z(t))C\Omega C(t, t_\mathrm{w}) - \int \mathrm{d}t'\, L(t, t')C\Omega C(t', t_\mathrm{w}) \tag{5.11}$$

$$= 2z(t)C\Omega C(t, t_\mathrm{w}) - \int \mathrm{d}t'\, L^{(1)}(t, t')C\Omega C(t', t_\mathrm{w}). \tag{5.12}$$

Note that we could use the same trick here as in (5.3), (5.5) to write $C\Omega C(t', t_\mathrm{w}) = -\left(\frac{1}{2}\partial_{t'} + z(t')\right)CC(t', t_\mathrm{w})$, with $CC(t, t_\mathrm{w})$ defined in the obvious manner as $CC = N^{-1}C_{ik}C_{ki} = \int(\mathrm{d}q)\omega C_{\mathbf{q}}^2$. However, this reduction from $C\Omega C$ to CC applies only for $t' > t_\mathrm{w}$, while for $t' < t_\mathrm{w}$ one would need to take a time derivative w.r.t. t_w instead of t'. This case distinction would make evaluation of (5.12) awkward, so we retain $C\Omega C$ here and below.

The third contribution to C_E is obtained by simply swapping the roles of t and t_w in (5.12) and remembering that $C\Omega C$ is symmetric in its time arguments,

$$4C_E^{(3)} = \frac{2}{N^{3/2}}\Omega_{ij}\Omega_{kl}\langle s_i(t)s_j(t)s_k(t_\mathrm{w})r_l(t_\mathrm{w})\rangle' \tag{5.13}$$

$$= 2z(t_\mathrm{w})C\Omega C(t, t_\mathrm{w}) - \int \mathrm{d}t'\, L^{(1)}(t_\mathrm{w}, t')C\Omega C(t, t'). \tag{5.14}$$

The fourth and last contribution to C_E is, using again (4.11) and writing $\Delta(t'')$ and $\Delta(t''_\mathrm{w})$,

$$4C_E^{(4)} = \frac{4}{N^2}\Omega_{ij}\Omega_{kl}\langle s_i(t)r_j(t)s_k(t_\mathrm{w})r_l(t_\mathrm{w})\rangle' \tag{5.15}$$

$$= \frac{1}{N^3}\int \mathrm{d}t'\,\mathrm{d}t''\,\mathrm{d}t'_\mathrm{w}\,\mathrm{d}t''_\mathrm{w}\,\Omega_{ij}\Omega_{kl}R_{jm}(t, t')L(t', t'')R_{ln}(t_\mathrm{w}, t'_\mathrm{w})L(t'_\mathrm{w}, t''_\mathrm{w})$$
$$\times \left\langle s_i(t)s_m(t')s_o^2(t'')s_k(t_\mathrm{w})s_n(t'_\mathrm{w})s_p^2(t''_\mathrm{w})\right\rangle' \tag{5.16}$$

where it is understood that, because of subtractions which we have not written explicitly, pairings of s_o with itself and of s_p with itself are to be excluded. The only pairing that gives an overall $\mathcal{O}(1)$ contribution turns out to be $[im][kn][op][op]$ and gives the required three independent index groups (i, m; k, n; o, o, p, p) to cancel the $1/N^3$ prefactor, see figure 2. With the symmetry factor 2 for the possible swap of the $[op][op]$ pairings one gets

$$4C_E^{(4)} = 2\int \mathrm{d}t'\,\mathrm{d}t''\,\mathrm{d}t'_\mathrm{w}\,\mathrm{d}t''_\mathrm{w}\,\frac{1}{N}\Omega_{ij}R_{jm}(t, t')C_{im}(t, t')$$
$$\times \frac{1}{N}\Omega_{kl}R_{ln}(t_\mathrm{w}, t'_\mathrm{w})C_{kn}(t_\mathrm{w}, t'_\mathrm{w})L(t', t'')L(t'_\mathrm{w}, t''_\mathrm{w})\frac{1}{N}C_{op}^2(t'', t''_\mathrm{w}) \tag{5.17}$$

$$= \frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t''\,\mathrm{d}t'_\mathrm{w}\,\mathrm{d}t''_\mathrm{w}[(\partial_t + 2z(t))K(t, t') - \delta(t - t')]L(t', t'')$$
$$\times [(\partial_{t_\mathrm{w}} + 2z(t_\mathrm{w}))K(t_\mathrm{w}, t'_\mathrm{w}) - \delta(t_\mathrm{w} - t'_\mathrm{w})]L(t'_\mathrm{w}, t''_\mathrm{w})CC(t'', t''_\mathrm{w}). \tag{5.18}$$

Using (4.9), one can carry out two of the time integrations to get

$$4C_E^{(4)} = \frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t'_\mathrm{w}[2z(t)\delta(t-t') - L^{(1)}(t,t')][2z(t_\mathrm{w})\delta(t_\mathrm{w}-t'_\mathrm{w}) - L^{(1)}(t_\mathrm{w},t'_\mathrm{w})]CC(t',t'_\mathrm{w}). \tag{5.19}$$

5.2. *Energy response function*

To find the energy response, consider perturbing the Hamiltonian by a term $-h\delta(t-t_\mathrm{w})H = -(h/2)\delta(t-t_\mathrm{w})S_i\Omega_{ij}S_j$, where h is the field conjugate to the energy. The equation of motion in the presence of the perturbation is therefore

$$\partial_t S_i = -\Omega_{ij}S_j - (z(t) + h\Delta z(t))S_i + h\delta(t-t_\mathrm{w})\Omega_{ij}S_j \tag{5.20}$$

where $h\Delta z$ is now the change in the Lagrange multiplier induced by the perturbation. The fluctuating component of z of $\mathcal{O}(N^{-1/2})$ is in principle still present, but negligible compared to $h\Delta z$ for field strengths that are $\mathcal{O}(N^0)$. Inserting a corresponding expansion of the spins, $S_i = s_i + hr_i$, gives for s_i the unperturbed equation of motion and for the perturbed component r_i

$$\partial_t r_i = -\Omega_{ij}r_j - z(t)r_i - \Delta z(t)s_i + \delta(t-t_\mathrm{w})\Omega_{ij}s_j. \tag{5.21}$$

The solution of this is

$$r_i(t) = R_{ik}(t,t_\mathrm{w})\Omega_{kl}s_l(t_\mathrm{w}) - \int_{t_\mathrm{w}}^t \mathrm{d}t'\,R_{ik}(t,t')\Delta z(t')s_k(t'). \tag{5.22}$$

One now needs to determine Δz. This can be done by noting that the normalized length of S_i is

$$\frac{1}{N}\sum_i S_i^2 = \frac{1}{N}\sum_i s_i^2 + 2h\frac{1}{N}\sum_i s_i r_i + \mathcal{O}(h^2). \tag{5.23}$$

The change to first order in h must vanish to preserve the spherical constraint, giving the condition $(1/N)\sum_i\langle s_i r_i\rangle = 0$ or, using (5.22),

$$\frac{1}{N}R_{ik}(t,t_\mathrm{w})\Omega_{kl}C_{il}(t,t_\mathrm{w}) = \int_{t_\mathrm{w}}^t \mathrm{d}t'\,\frac{1}{N}R_{ik}(t,t')C_{ik}(t,t')\Delta z(t'). \tag{5.24}$$

In the integrand one recognizes definition (4.8) of K, so that one can write the solution of this as

$$\Delta z(t) = \int \mathrm{d}t'\,L(t,t')R\Omega C(t',t_\mathrm{w}), \tag{5.25}$$

with obvious notation for $R\Omega C$.

Now, we can find the change in the energy, $1/(2N)\langle S_i\Omega_{ij}S_j\rangle$, which is given by $(h/N)\langle r_i\Omega_{ij}s_j\rangle$ to linear order in h. Dividing by h and using (5.22) then gives the energy response function

$$R_E(t,t_\mathrm{w}) = \frac{1}{N}\langle r_i\Omega_{ij}s_j\rangle \tag{5.26}$$

$$= \frac{1}{N}R_{ik}(t,t_\mathrm{w})\Omega_{kl}\Omega_{ij}C_{jl}(t,t_\mathrm{w}) - \int \mathrm{d}t'\,\mathrm{d}t''\,R\Omega C(t,t')L(t',t'')R\Omega C(t'',t_\mathrm{w}). \tag{5.27}$$

One can eliminate one of the time integrals by using (5.3), being careful to remove the unwanted contribution from differentiating the step discontinuity in $R\Omega C(t,t_\mathrm{w})$. Using also

(4.9) and (4.12) then gives

$$2R_E = (-\partial_t - 2z(t))R\Omega C(t,t_w) + \delta(t-t_w)R\Omega C(t_w^+,t_w)$$
$$-\int dt'\,dt''[(-\partial_t - 2z(t))K(t,t') + \delta(t-t')]L(t',t'')R\Omega C(t'',t_w) \tag{5.28}$$
$$= -\int dt'\,L(t,t')R\Omega C(t',t_w) + \delta(t-t_w)R\Omega C(t_w^+,t_w). \tag{5.29}$$

5.3. Equilibrium

Above, we derived general expressions for the energy two-time correlation and response, in terms of the known correlation and response functions for the Gaussian spins which in turn determine K and L. Before looking at non-equilibrium, we briefly consider the equilibrium situation; even here the results for the dynamics are new as far as we are aware.

For the response function one uses that at equilibrium $R\Omega^a C(t) = \int(dq)\,e^{-2(z_{eq}+\omega)t}\omega^a[T/(z_{eq}+\omega)]$ for $a=1,2$. Here, we have retained a possible nonzero equilibrium value z_{eq} of the Lagrange multiplier, to include the case of equilibrium at $T \neq T_c$. Inserting into (5.27) and taking LTs gives

$$\hat{R}_E^{eq}(s) = \int(dq)\frac{T\omega^2}{(z_{eq}+\omega)[s+2(z_{eq}+\omega)]} - \frac{1}{\hat{K}_{eq}(s)}\left(\int(dq)\frac{T\omega}{(z_{eq}+\omega)[s+2(z_{eq}+\omega)]}\right)^2 \tag{5.30}$$

where $\hat{K}_{eq}$ is generalized from (4.22) to

$$\hat{K}_{eq}(s) = T\int(dq)\frac{1}{(z_{eq}+\omega)[s+2(z_{eq}+\omega)]}. \tag{5.31}$$

Using the spherical constraint condition $T\int(dq)(z_{eq}+\omega)^{-1}=1$ one then shows, after a few lines of algebra, that

$$\hat{R}_E^{eq}(s) = \frac{1}{4}\left(s+2T-\frac{1}{\hat{K}_{eq}(s)}\right) = \frac{1}{4}\hat{L}_{eq}^{(2)}(s). \tag{5.32}$$

Remarkably, therefore, the equilibrium energy response function $R_E^{eq}(t)$ is directly proportional to the inverse kernel $L_{eq}^{(2)}(t)$. Its asymptotics are then given by (4.23). The long-time equilibrium susceptibility which encodes the response to a step change in the field is, using $\hat{K}_{eq}(0) = \int(dq)T/[2(z_{eq}+\omega)^2]$ and the generalization of (4.25) to $T\neq T_c$,

$$\chi_E^{eq} = \int_0^\infty dt\,R_E^{eq}(t) = \frac{1}{4}\hat{L}_{eq}^{(2)}(0) = \frac{T}{2}\left(1-\frac{1}{\int(dq)[T/(z_{eq}+\omega)]^2}\right). \tag{5.33}$$

It is easily shown that this is consistent with the known result for the temperature dependence of the equilibrium energy, $E = \langle H\rangle = \int(dq)\frac{1}{2}\omega[T/(z_{eq}+\omega)]$: one confirms $\chi_E^{eq} = T\,dE/dT$ as it should be. The factor T arises because our field h is introduced via $H\to H-hH=(1-h)H$ and so corresponds to a temperature change of $T/(1-h)-T=hT$ to linear order in h. The inclusion of subleading non-Gaussian fluctuations is crucial for achieving this consistency; as discussed at the beginning of section 4, the Gaussian theory does not even give the same answers for the fluctuations of H in its two representations as a bond or spin product observable. The same phenomenon occurs in a purely static calculation of the energy fluctuations and response [19].

The temperature dependence of the susceptibility (5.33) deserves some comment. As T approaches T_c from above, one has $z_{eq} \to 0$. For $d < 4$, the denominator of the second term in (5.33) then diverges, and χ_E^{eq}/T smoothly approaches the value $1/2$ and remains constant for $T < T_c$. For $d > 4$, the denominator has a finite limit for $z_{eq} \to 0$. This produces the well-known discontinuity in χ_E^{eq}/T at $T = T_c$, since for $T < T_c$ the second term in (5.33) again does not contribute. To see this explicitly, one notes that for $T < T_c$ the $\omega = 0$ term in the spherical constraint condition, with its weight $1/N$, has to be treated separately:

$$T\left(\int (\mathrm{d}q)\frac{1}{z_{eq}+\omega} + \frac{1}{Nz_{eq}}\right) = 1. \tag{5.34}$$

For $z_{eq} \to 0$ (on the scale $\mathcal{O}(N^0)$), the first integral is $\int(\mathrm{d}q)1/\omega = 1/T_c$ and so $z_{eq} = (1/N)TT_c/(T_c - T)$. This then gives for the denominator integral in (5.33)

$$\int (\mathrm{d}q)\frac{T^2}{(z_{eq}+\omega)^2} + \frac{T^2}{Nz_{eq}^2} \approx N\frac{(T_c-T)^2}{T_c^2} \tag{5.35}$$

which diverges for $N \to \infty$ at $T < T_c$ as claimed.

An interesting question is how the discontinuity at $T = T_c$ in $d > 4$ of the susceptibility χ_E^{eq}, i.e. the *integrated* response, is related to the temperature variation of the *time-dependent* $R_E^{eq}(t)$. In (5.31) one notes that, for $T < T_c$, $\hat{K}_{eq}(s)$ acquires a distinct contribution from the $\omega = 0$ mode

$$\hat{K}_{eq}(s) = \int (\mathrm{d}q)\frac{T}{\omega(s+2\omega)} + \frac{1}{N}\frac{T}{z_{eq}(s+2z_{eq})} \tag{5.36}$$

$$= \int (\mathrm{d}q)\frac{T}{\omega(s+2\omega)} + \frac{T_c-T}{T_c s} \tag{5.37}$$

where in the second line we have neglected $z_{eq} = \mathcal{O}(1/N)$ against s. From (5.32), one sees that the additional contribution to $\hat{K}_{eq}(s)$ produces an extra pole in $\hat{R}_E^{eq}(s)$, which for small $(T_c - T)/T_c$ is located at $s = 1/t_0$ with $t_0 = [TT_c/4(T_c - T)]\int(\mathrm{d}q)\omega^{-2}$. Transforming to the time domain, one finds that this gives an extra contribution of $\sim 1/t_0 \exp(-t/t_0)$ to $R_E^{eq}(t)$. Crucially, this has a finite integral even in the limit $T \to T_c$, where t_0 diverges, and the appearance of this term causes the discontinuity in χ_E^{eq} at $T = T_c$. Translating these results to the time-dependent susceptibility

$$\chi_E^{eq}(t) = \int_0^t \mathrm{d}t'\, R_E^{eq}(t') \tag{5.38}$$

one has that, for fixed finite t, $\chi_E^{eq}(t)$ depends smoothly on temperature around T_c. For large t, $\chi_E^{eq}(t)$ approaches a plateau value equal to the equilibrium susceptibility at $T = T_c^+$; the approach to this plateau is as $\sim t^{(4-d)/2}$. For $T < T_c$, however, $\chi_E^{eq}(t)$ eventually increases further on a diverging time scale $t \sim t_0 \sim 1/(T_c - T)$ to approach a higher plateau value given by the susceptibility at $T = T_c^-$. By FDT (see below), the energy correlation function correspondingly shows a power-law decay to a plateau for $t \ll t_0$, from which it falls to zero only for $t > t_0$.

A final check on our results is that in equilibrium the energy response and correlation function should be related by the usual FDT. This is indeed the case. One combines the results (5.5), (5.12), (5.14), (5.19) for the constituent parts of the correlation function and decouples all the convolution integrals using temporal Fourier transforms. It is easiest to do this starting from expressions where all factors of Ω have been preserved, e.g. for $C_E^{(1)}$ one uses (5.2) rather than (5.5). After some straightforward but lengthy algebra, one then indeed finds the Fourier domain version of equilibrium FDT, $\int_{-\infty}^{\infty} \mathrm{d}t\, C_E^{eq}(t)\,\mathrm{e}^{-\mathrm{i}\nu t} = T\big[\hat{R}_E^{eq}(-\mathrm{i}\nu) - \hat{R}_E^{eq}(\mathrm{i}\nu)\big]/(\mathrm{i}\nu)$.

6. Energy correlation and response: non-equilibrium, $d > 4$

We now evaluate the behaviour of the energy correlation and response for the out-of-equilibrium dynamics after a quench to criticality. For the correlation function, we need the long-time behaviour of

$$C\Omega C(t, t_w) = \int (\mathrm{d}q) \frac{T_c^2}{\omega^2} \mathcal{F}_C^2(\omega t_w) \frac{g(t_w)}{g(t)} \mathrm{e}^{-2\omega(t-t_w)} \omega. \tag{6.1}$$

At equilibrium, where $\mathcal{F}_C = 1$ and $g(t) = \text{const}$, this gives $T_c K_{eq}(t - t_w)$. In the non-equilibrium case, one gets a correction factor which becomes relevant in the ageing regime, where $t - t_w \sim t_w \gg 1$:

$$C\Omega C(t, t_w) = T_c K_{eq}(t - t_w) \mathcal{F}_{C\Omega C}(t/t_w) \tag{6.2}$$

$$\mathcal{F}_{COC}(x) = \left(\frac{t}{t_w}\right)^{\kappa} \frac{\int \mathrm{d}w\, w^{(d-4)/2}\, \mathrm{e}^{-2(x-1)w} \mathcal{F}_C^2(w)}{\int \mathrm{d}w\, w^{(d-4)/2}\, \mathrm{e}^{-2(x-1)w}}. \tag{6.3}$$

This is valid for all dimensions $d > 2$, but in the rest of this section we focus on the case $d > 4$, where $\kappa = 0$. The regime $2 < d < 4$ is more complicated and treated separately in the next section.

Before assembling the four parts of our expression for $C_E(t, t_w)$, it is useful to have a guide on what to expect overall. As discussed above, the equilibrium energy fluctuations remain finite at criticality. One therefore expects that, in the short-time regime $t - t_w \sim \mathcal{O}(1)$, $C_E(t, t_w)$ will decay as in equilibrium. From (4.23), (5.32) and FDT, it follows that this decay becomes a power law, $\sim(t - t_w)^{(4-d)/2}$ as $t - t_w$ becomes large. Aging effects should then appear when $t - t_w \sim t_w$ and manifest themselves through a scaling function of t/t_w. Overall, one expects in the ageing regime

$$C_E(t, t_w) = (t - t_w)^{(4-d)/2} \times [\text{scaling function of } t/t_w] \tag{6.4}$$

and this is indeed what we will find. Consider now $C_E^{(1)}$, for which we obtained in (5.5) that $4C_E^{(1)} = -(\partial_t + 2z(t))C\Omega C(t, t_w)$. Now from (2.7) and the fact that $g(t) \to \text{const}$ for $t \to \infty$ and $d > 4$, it follows that $z(t) = g'(t)/[2g(t)]$ decreases more quickly than $1/t$. Using also (6.2), we see that the $z(t)$ term is negligible for large times and that $4C_E^{(1)}$ decays as $(t - t_w)^{-d/2}$ times an ageing correction. This is negligible compared to (6.4), so we do not need to analyse this contribution further.

For $C_E^{(2)}$, we have from (5.12)

$$4C_E^{(2)} = 2z(t)C\Omega C(t, t_w) - 2T_c C\Omega C(t, t_w) + \int \mathrm{d}t'\, L^{(2)}(t, t')C\Omega C(t', t_w). \tag{6.5}$$

Because of the factor $C\Omega C(t', t_w)$, it is useful to split the t'-integral into the regimes $t' > t_w$ and $t' < t_w$. The first regime gives

$$T_c \int_{t_w} \mathrm{d}t'\, L_{eq}^{(2)}(t - t')K_{eq}(t' - t_w)\mathcal{F}_L(t/t')\mathcal{F}_{C\Omega C}(t'/t_w). \tag{6.6}$$

As in the analysis of (4.27) one can now argue that, for large $t - t_w$, the integral will be dominated by the regions $t' \approx t$ and $t' \approx t_w$. The weights contributed by these regions are $K_{eq}(t - t_w) \int_0^\infty \mathrm{d}t'\, L_{eq}^{(2)}(t')$ and $L_{eq}^{(2)}(t - t_w) \int_0^\infty \mathrm{d}t'\, K_{eq}(t')$, respectively; the integrals are both finite, so that both terms scale as $(t - t_w)^{(2-d)/2}$. These weights are then multiplied by the relevant values of scaling functions in the integrand of (6.6). The overall result is smaller by

$\sim 1/(t-t_{\rm w})$ than the leading term (6.4) in C_E and can be neglected. The $t' < t_{\rm w}$ part of the integral from (6.5) is

$$T_{\rm c}\int^{t_{\rm w}} {\rm d}t'\, L^{(2)}_{\rm eq}(t-t')K_{\rm eq}(t_{\rm w}-t')\mathcal{F}_L(t/t')\mathcal{F}_{C\Omega C}(t_{\rm w}/t'). \tag{6.7}$$

The factor $K_{\rm eq}(t_{\rm w}-t')$ concentrates the weight of the integrand near $t' = t_{\rm w}$, so that for $t-t_{\rm w}$ and $t_{\rm w} \gg 1$ all other factors can be replaced by their values at $t' = t_{\rm w}$, giving the result $T_{\rm c}\hat{K}_{\rm eq}(0)L^{(2)}_{\rm eq}(t-t_{\rm w})\mathcal{F}_L(x)$. This scales as $(t-t_{\rm w})^{(2-d)/2}$ times a scaling function of x and so is also negligible compared to the leading contribution (6.4). From (6.2), the second term on the rhs of (6.5) has the same subleading scaling and the first term is even smaller. Overall, $C_E^{(2)}$ therefore gives a subleading contribution to C_E in the ageing regime. By very similar arguments one shows that $C_E^{(3)}$ from (5.14) is also subleading.

In the ageing regime, we therefore only need to consider $C_E^{(4)}$ from (5.19). The long-time behaviour of the function $CC(t, t_{\rm w})$ is easily worked out as

$$CC(t, t_{\rm w}) = CC_{\rm eq}(t-t_{\rm w})\mathcal{F}_{CC}(t/t_{\rm w}) \tag{6.8}$$

$$CC_{\rm eq}(t-t_{\rm w}) = \int ({\rm d}q)\frac{T_{\rm c}^2}{\omega^2}\,{\rm e}^{-2\omega(t-t_{\rm w})} = 2T_{\rm c}\int_{t-t_{\rm w}}^{\infty} {\rm d}t'\, K_{\rm eq}(t') \sim (t-t_{\rm w})^{(4-d)/2} \tag{6.9}$$

$$\mathcal{F}_{CC}(x) = \frac{\int {\rm d}w\, w^{(d-6)/2}\,{\rm e}^{-2(x-1)w}\mathcal{F}_C^2(w)}{\int {\rm d}w\, w^{(d-6)/2}\,{\rm e}^{-2(x-1)w}} \tag{6.10}$$

where the scaling function $\mathcal{F}_{CC}(x) \sim x^{-2}$ for $x \gg 1$. Now consider the t'-integral from (5.19),

$$I(t, t'_{\rm w}) = \int {\rm d}t'[(2z(t) - 2T_{\rm c})\delta(t-t') + L^{(2)}(t, t')]CC(t', t'_{\rm w}). \tag{6.11}$$

The equilibrium part $L^{(2)}_{\rm eq}(t-t')$ of $L^{(2)}(t, t')$ ensures that this has its mass concentrated near $t' = t$. Because we are looking at the ageing regime, we have $t - t'_{\rm w} (> t - t_{\rm w}) \gg 1$ and (6.9) then shows that the function $CC(t', t'_{\rm w})$ is slowly varying near $t' = t$. It can thus be replaced by its value there, $CC(t, t'_{\rm w})$, to give to leading order

$$I(t, t'_{\rm w}) = (2z(t) - 2T_{\rm c} + \hat{L}^{(2)}_{\rm eq}(0))CC(t, t'_{\rm w}) = \frac{1}{\hat{K}_{\rm eq}(0)}CC(t, t'_{\rm w}) \tag{6.12}$$

where in the last step we have used $z(t) \ll 1$. One might suspect that in the t'-integral of (6.11), the region around $t' \approx t'_{\rm w}$ also contributes, since this is where $CC(t', t'_{\rm w})$ is largest. However, it can be shown that this region only gives a subleading contribution compared to (6.12).

In terms of $I(t, t'_{\rm w})$, we can write $C_E^{(4)}$ using (5.19) as

$$4C_E^{(4)} = \frac{1}{2}\int {\rm d}t'_{\rm w}[(2z(t_{\rm w}) - 2T_{\rm c})\delta(t_{\rm w}-t'_{\rm w}) + L^{(2)}(t_{\rm w}, t'_{\rm w})]I(t, t'_{\rm w}). \tag{6.13}$$

Again, the integral is dominated by the region $t'_{\rm w} \approx t_{\rm w}$ because of the factor $L^{(2)}_{\rm eq}(t_{\rm w}-t'_{\rm w})$ and we thus get our final expression

$$4C_E^{(4)} = \frac{1}{2\hat{K}_{\rm eq}(0)}I(t, t_{\rm w}) = \frac{1}{2\hat{K}^2_{\rm eq}(0)}CC(t, t_{\rm w}). \tag{6.14}$$

As argued above, in the ageing regime the full energy correlator will be identical to this.

We can now turn to the evaluation of the out-of-equilibrium response function. We will not write explicitly the second term in (5.29), which just removes the unwanted $\delta(t-t_{\rm w})$ contribution arising from the derivative applied to the first term's step discontinuity. Thus,

$$2R_E = (-\partial_t - 2T_{\rm c})R\Omega C(t, t_{\rm w}) + \int {\rm d}t'\, L^{(2)}(t, t')R\Omega C(t', t_{\rm w}). \tag{6.15}$$

The long-time scaling of the function $R\Omega C$ is

$$R\Omega C(t,t_{\rm w}) = R\Omega C_{\rm eq}(t-t_{\rm w})\mathcal{F}_{R\Omega C}(t/t_{\rm w}) \tag{6.16}$$

$$R\Omega C_{\rm eq}(t-t_{\rm w}) = T_{\rm c}\int ({\rm d}q)\,{\rm e}^{-2\omega(t-t_{\rm w})} = T_{\rm c}f(t-t_{\rm w}) \sim (t-t_{\rm w})^{-d/2} \tag{6.17}$$

$$\mathcal{F}_{R\Omega C}(x) = \frac{\int {\rm d}w\, w^{(d-2)/2}\,{\rm e}^{-2(x-1)w}\mathcal{F}_C(w)}{\int {\rm d}w\, w^{(d-2)/2}\,{\rm e}^{-2(x-1)w}}. \tag{6.18}$$

The integrand in (6.15) has its mass concentrated near $t' = t$, from the factor $L^{(2)}_{\rm eq}(t-t')$, and near $t' = t_{\rm w}$, from the factor $R\Omega C_{\rm eq}(t'-t_{\rm w})$. This gives

$$2R_E = (-\partial_t - 2T_{\rm c})R\Omega C(t,t_{\rm w}) + \hat{L}^{(2)}_{\rm eq}(0)R\Omega C(t,t_{\rm w}) + \tfrac{1}{2}L^{(2)}_{\rm eq}(t-t_{\rm w})\mathcal{F}_L(t/t_{\rm w}) \tag{6.19}$$

using that $\int_0^\infty {\rm d}t'\, R\Omega C_{\rm eq}(t') = \frac{1}{2}2T_{\rm c}\int_0^\infty {\rm d}t'\, f(t') = \frac{1}{2}K_{\rm eq}(0) = \frac{1}{2}$ (see after (4.13)). The last term in (6.19) scales as $(t-t_{\rm w})^{(2-d)/2}$ times a function of $x = t/t_{\rm w}$, while the other terms scale at most as $(t-t_{\rm w})^{-d/2}$ and so are subdominant. This gives the final ageing regime expression

$$R_E = \tfrac{1}{4}L^{(2)}_{\rm eq}(t-t_{\rm w})\mathcal{F}_L(t/t_{\rm w}) = \tfrac{1}{4}L^{(2)}(t,t_{\rm w}). \tag{6.20}$$

Interestingly, the simple relation (5.32) between the energy response and $L^{(2)}$ therefore holds not only at equilibrium but also in the ageing regime of the non-equilibrium dynamics and thus overall across the entire long-time regime.

We can now, finally, find the FDR for the energy correlation and response. In the ageing regime of interest here, it is useful to rewrite the response function (6.20) as

$$R_E(t,t_{\rm w}) = \frac{1}{4\hat{K}^2_{\rm eq}(0)}K_{\rm eq}(t-t_{\rm w})\mathcal{F}_K(x) = \frac{1}{4\hat{K}^2_{\rm eq}(0)}K(t,t_{\rm w}) \tag{6.21}$$

using (4.24) and $\mathcal{F}_L = \mathcal{F}_K$. Recalling definition (4.8), and the fact that C_E is given by (6.14) to leading order in the ageing limit, we then obtain

$$X_E(t,t_{\rm w}) = \frac{T_{\rm c}R_E(t,t_{\rm w})}{C'_E(t,t_{\rm w})} = \frac{\int({\rm d}q)T_{\rm c}R_{\mathbf{q}}C_{\mathbf{q}}}{\int({\rm d}q)C'_{\mathbf{q}}C_{\mathbf{q}}}. \tag{6.22}$$

Remarkably, this is in fact *identical* to the FDR (3.31) for large blocks of spin product observables. In particular, X_E tends to the limit value $X^\infty = 1/2$ for $t \gg t_{\rm w}$, and this is identical to the one we found for spin and bond observables: for $d > 4$, all observables we have considered lead to a unique value of $X^\infty = 1/2$.

The FD plot for the energy has a limiting pseudo-equilibrium form; this follows from (6.9) and (6.14), which show that $C_E(t,t_{\rm w})$ has decayed to a small value $\sim t_{\rm w}^{(4-d)/2}$ at the point where ageing effects begin to appear. More generally, the correlation function scales as $C_E \sim (t-t_{\rm w})^{(4-d)/2}$ for $x \approx 1$, and for $x \gg 1$, where $\mathcal{F}_{CC}(x) \sim x^{-2}$ from (6.10), as $C_E \sim t^{(4-d)/2}x^{-2} = t_{\rm w}^2 t^{-d/2}$.

It should be pointed out that while the FDR for the energy matches that for blocks of product observables at long times, the correlation and response functions themselves do not. This follows from the nontrivial proportionality factors $1/\hat{K}^2_{\rm eq}(0)$ in (6.14) and (6.21). The latter are required to match the limiting behaviour for $t - t_{\rm w} \ll t_{\rm w}$ of the ageing regime results to the asymptotics of the equilibrium results for $t - t_{\rm w} \gg 1$. Indeed, by combining the effective equilibrium behaviour for $t - t_{\rm w} = \mathcal{O}(1)$ with the above ageing expressions, we can write

$$C_E(t,t_{\rm w}) = C_E^{\rm eq}(t-t_{\rm w})\mathcal{F}_{CC}(x) \tag{6.23}$$

$$R_E(t,t_{\rm w}) = R_E^{\rm eq}(t-t_{\rm w})\mathcal{F}_K(x) \tag{6.24}$$

and these are now valid throughout the long-time regime, i.e. for large t_w but independently of whether $t - t_\mathrm{w}$ is large or not. As promised they match with the ageing expressions for $1 \ll \Delta t \ll t_\mathrm{w}$. For the response this is obvious from (5.32). For C_E, (4.24) and (5.32) show that $-\partial_{\Delta t} C_E^{\mathrm{eq}}(\Delta t) = (T_\mathrm{c}/4) L_{\mathrm{eq}}^{(2)}(\Delta t) \approx \left[T_\mathrm{c}/4\hat{K}_{\mathrm{eq}}^2(0)\right] K_{\mathrm{eq}}(\Delta t)$ for large Δt; from (6.9) this agrees with the corresponding derivative of the result in (6.14), $-\partial_{\Delta t} CC_{\mathrm{eq}}(\Delta t)/\left[8\hat{K}_{\mathrm{eq}}^2(0)\right] = \left[2T_\mathrm{c}/8\hat{K}_{\mathrm{eq}}^2(0)\right] K_{\mathrm{eq}}(\Delta t)$. Note from the discussion after (5.38) that the function $C_E^{\mathrm{eq}}(\Delta t)$ discontinuously acquires an additive constant (non-decaying) part as T crosses T_c from above in $d > 4$. What we mean in (6.23) is the limiting form for $T \searrow T_\mathrm{c}$ *from above*, which does not contain this plateau. That this is the correct choice can be seen as follows: the non-decaying part of $C_E^{\mathrm{eq}}(\Delta t)$ at $T = T_\mathrm{c}^-$ arises from the fluctuations of the $\mathbf{q} = \mathbf{0}$ Fourier mode, i.e. of the magnetization, which are larger by $\mathcal{O}(N)$ than those of the other Fourier modes. In the context of our non-equilibrium calculation, all Fourier modes have fluctuations of comparable order (in system size), so that this contribution is absent; it would appear only for times t_w that scale with system size N.

Finally, the long-time expressions (6.23), (6.24) also show that an energy FD plot would have a pseudo-equilibrium form at long times, because, e.g., in $C_E(t, t_\mathrm{w})$ the equilibrium factor has already decayed to $\sim t_\mathrm{w}^{(4-d)/2}$ of its initial value when ageing effects appear around $\Delta t \sim t_\mathrm{w}$.

7. Energy correlation and response: non-equilibrium, $d < 4$

In dimension $d < 4$, the evaluation of the energy correlation function in the ageing regime is somewhat more complicated. One can nevertheless show that, as before, the dominant contribution to C_E is $C_E^{(4)}$, with the other three terms being subleading. We thus again need to consider the function $I(t', t_\mathrm{w})$ defined in (6.11) and then work out C_E using (6.13). One can clearly see that the approach leading above to (6.12) no longer works: in $d < 4$, $\hat{L}_{\mathrm{eq}}^{(2)}(0) = 2T_\mathrm{c}$, so the leading terms in (6.12) actually cancel. To treat this cancellation accurately, we use (4.20) to write the coefficient $2T_\mathrm{c}$ in the following way:

$$2T_\mathrm{c} = -\frac{K_{\mathrm{eq}}'(t)}{K_{\mathrm{eq}}(t)} + \int \mathrm{d}t'\, \frac{K_{\mathrm{eq}}(t')}{K_{\mathrm{eq}}(t)} L_{\mathrm{eq}}^{(2)}(t - t'). \tag{7.1}$$

This allows us to express (6.11) as

$$\begin{aligned} I(t, t_\mathrm{w}') = {} & \left[2z(t) + \frac{K_{\mathrm{eq}}'(t)}{K_{\mathrm{eq}}(t)}\right] CC(t, t_\mathrm{w}') \\ & + \int_0^t \mathrm{d}t'\, L_{\mathrm{eq}}^{(2)}(t - t') \left[\mathcal{F}_L(t/t') CC(t', t_\mathrm{w}') - \frac{K_{\mathrm{eq}}(t')}{K_{\mathrm{eq}}(t)} CC(t, t_\mathrm{w}')\right]. \end{aligned} \tag{7.2}$$

To make progress, we need the behaviour of CC for $d < 4$. For long times, we have

$$CC(t, t_\mathrm{w}) = \int (\mathrm{d}q) C_{\mathbf{q}}^2(t, t_\mathrm{w}) = T_\mathrm{c}^2 \frac{g(t_\mathrm{w})}{g(t)} \int (\mathrm{d}q) \omega^{-2} \mathcal{F}_C^2(\omega t_\mathrm{w})\, \mathrm{e}^{-2\omega(t - t_\mathrm{w})}. \tag{7.3}$$

The equal-time value thus scales as

$$CC(t, t) \sim \int \mathrm{d}\omega\, \omega^{d/2-3} \mathcal{F}_C^2(\omega t) = t^{(4-d)/2} \int \mathrm{d}w\, w^{(d-6)/2} \mathcal{F}_C^2(w) \tag{7.4}$$

hence $CC(t, t) = \gamma_d t^{(4-d)/2}$ with some constant γ_d. A scaling function is then obtained if we normalize $CC(t, t_\mathrm{w})$ by this equal-time value:

$$\frac{CC(t, t_\mathrm{w})}{CC(t, t)} = \frac{g(t_\mathrm{w})}{g(t)} \frac{t_\mathrm{w}^{(4-d)/2}}{t^{(4-d)/2}} \frac{\int \mathrm{d}w\, w^{(d-6)/2} \mathcal{F}_C^2(w)\, \mathrm{e}^{-2w(t - t_\mathrm{w})/t_\mathrm{w}}}{\int \mathrm{d}w\, w^{(d-6)/2} \mathcal{F}_C^2(w)}. \tag{7.5}$$

This is a function $\mathcal{G}(x)$ of $x = t/t_w$; note that the first two fractions cancel since $g(t) \sim t^{-\kappa} = t^{(d-4)/2}$. So far this scaling expression holds for $t > t_w$. To be able to use it also for non-ordered times, note that for $t < t_w$,

$$\frac{CC(t,t_w)}{CC(t,t)} = \left(\frac{t_w}{t}\right)^{(4-d)/2} \frac{CC(t_w,t)}{CC(t_w,t_w)} = x^{(d-4)/2}\mathcal{G}(t_w/t). \tag{7.6}$$

So, we have overall

$$\frac{CC(t,t_w)}{CC(t,t)} = \mathcal{G}(t/t_w), \qquad \mathcal{G}(x) = \begin{cases} \dfrac{\int \mathrm{d}w\, w^{(d-6)/2}\mathcal{F}_C^2(w)\,\mathrm{e}^{-2(x-1)w}}{\int \mathrm{d}w\, w^{(d-6)/2}\mathcal{F}_C^2(w)} & \text{for} \quad x \geqslant 1 \\ x^{(d-4)/2}\mathcal{G}(1/x) & \text{for} \quad x \leqslant 1. \end{cases} \tag{7.7}$$

One easily works out the asymptotics of $\mathcal{G}$: $\mathcal{G}(x) \sim x^{-d/2}$ for $x \to \infty$, $\mathcal{G}(x) \sim x^{d-2}$ for $x \to 0$. For $|x-1| \ll 1$, on the other hand, $1 - \mathcal{G}(x) \sim |x-1|^{(4-d)/2}$. This corresponds to $CC(t,t) - CC(t,t_w) \sim |t-t_w|^{(4-d)/2}$ for $1 \ll |t-t_w| \ll t_w$. (The behaviour of this difference for smaller time intervals $t - t_w = \mathcal{O}(1)$ is not captured accurately by the scaling form (7.7), but integrals over this regime contribute only subleading corrections to the results below.)

We can now insert the scaling form (7.7) of CC into (7.2). The non-integral terms turn out to be subleading (see below), so

$$I(t,t_w') = \int \mathrm{d}t'\, L_{\mathrm{eq}}^{(2)}(t-t')\left[\mathcal{F}_L(t/t')CC(t',t')\mathcal{G}(t'/t_w') - \frac{K_{\mathrm{eq}}(t')}{K_{\mathrm{eq}}(t)}CC(t,t)\mathcal{G}(t/t_w')\right] \tag{7.8}$$

$$= CC(t,t)\int \mathrm{d}t'\, L_{\mathrm{eq}}^{(2)}(t-t')\left[\frac{t'}{t}\mathcal{G}(t'/t_w') - \frac{K_{\mathrm{eq}}(t')}{K_{\mathrm{eq}}(t)}\mathcal{G}(t/t_w')\right] \tag{7.9}$$

where we have used that, from (4.18) and (4.31), $\mathcal{F}_L(t/t')CC(t',t')/CC(t,t) = (t'/t)^{(d-2)/2}(t'/t)^{(4-d)/2} = t'/t$. The remaining 'subtracted' integral now no longer has its mass concentrated near $t' = t$ because the terms in square brackets give a factor $t - t'$ there. The whole integration range contributes, so that we can replace $L_{\mathrm{eq}}^{(2)}(t-t')$ by its asymptotic form (4.23), $L_{\mathrm{eq}}^{(2)}(t-t') \approx \lambda_d(t-t')^{(d-6)/2}$ with a d-dependent constant λ_d. Similarly, the ratio $K_{\mathrm{eq}}(t')/K_{\mathrm{eq}}(t)$ can be replaced by $(t'/t)^{(2-d)/2}$ to leading order. Scaling the integration variable as $y = t'/t_w'$ then gives

$$I(t,t_w') = I(x') = \gamma_d\lambda_d x'^{-2}\int_0^{x'} \mathrm{d}y(1-y/x')^{(d-6)/2}y[\mathcal{G}(y) - y^{-d/2}\mathcal{G}(x')x'^{d/2}]. \tag{7.10}$$

This shows that in the ageing regime I depends on $x' = t/t_w'$ only. One can now also see that the neglected terms from (7.2) are indeed subleading: they scale as $t^{-1}CC(t,t_w') \sim t^{(2-d)/2}\mathcal{G}(t/t_w')$.

We can now proceed to simplifying (6.13) in the ageing regime. Using the same subtraction method as above, and remembering that $C_E = C_E^{(4)}$ to leading order, one finds by analogy with (7.2)

$$8C_E(t,t_w) = \int \mathrm{d}t_w'\, L_{\mathrm{eq}}^{(2)}(t_w-t_w')\left[\mathcal{F}_L(t_w/t_w')I(t,t_w') - \frac{K_{\mathrm{eq}}(t_w')}{K_{\mathrm{eq}}(t_w)}I(t,t_w)\right]. \tag{7.11}$$

Here, subleading terms similar to those in the first line of (7.2) have already been neglected. With the scaled integration variable $y = t_w'/t_w$, and using the asymptotic forms of $L_{\mathrm{eq}}^{(2)}(t_w - t_w')$ and $K_{\mathrm{eq}}(t_w')$ as in (7.10), one gets

$$8C_E(t,t_w') = \lambda_d t_w^{(d-4)/2}\int_0^1 \mathrm{d}y(1-y)^{(d-6)/2}[y^{(d-2)/2}I(x/y) - y^{(2-d)/2}I(x)]. \tag{7.12}$$

This shows that C_E scales as $t_{\mathrm{w}}^{(d-4)/2}$ times a function of $x = t/t_{\mathrm{w}}$. It is difficult to work out the whole functional dependence on x. We therefore focus below on the asymptotic behaviour for $x \to \infty$, which gives the asymptotic FDR. First, though, we check that in the limit $x \to 1$, where ageing effects are unimportant, our expression matches with the equilibrium result $C_E^{\mathrm{eq}}(t - t_{\mathrm{w}})$. From (5.32), the latter behaves as $C_E^{\mathrm{eq}}(\Delta t) = (T_{\mathrm{c}}/4)\int_{\Delta t}^{\infty} \mathrm{d}t'\, L_{\mathrm{eq}}^{(2)}(t') \approx [\lambda_d T_{\mathrm{c}}/2(4-d)]\Delta t^{(d-4)/2}$ for large Δt. To compare with the prediction from (7.12), one uses that $I(x') \sim \ln(x'-1)$ for $x' \approx 1$; this follows from the behaviour of $\mathcal{G}(x)$ for $x \approx 1$. (Note that the proportionality constant in $I \sim \ln(x'-1)$ is *positive*, so that I itself is—in contrast to the case $d > 4$—*negative*.) Inserting into (7.12) one then finds that for $x \approx 1$ the integral scales as $(x-1)^{(d-4)/2}$. Overall, one gets $C_E(t, t_{\mathrm{w}}) \sim t_{\mathrm{w}}^{(d-4)/2}(x-1)^{(d-4)/2} = (t - t_{\mathrm{w}})^{(d-4)/2}$, consistent with the expectation from the equilibrium result. One can work out the prefactor of the power law and finds that this, too, agrees as it should.

Turning now to the behaviour of (7.12) for large x, we need the asymptotics of $I(x')$. The leading tail $\sim y^{-d/2}$ of $\mathcal{G}(y)$ is subtracted off in the expression in square brackets in (7.10), leaving as the next term $y^{-(d+2)/2}$. Even together with the additional factor y this makes the integral convergent at the upper end for $x' \to \infty$. In the limit we thus get $I(x') \approx -\alpha_d \gamma_d \lambda_d x'^{-2}$, with

$$\alpha_d = \int_0^{\infty} \mathrm{d}y\, y[g_d y^{-d/2} - \mathcal{G}(y)] \tag{7.13}$$

and $g_d = \lim_{x\to\infty} \mathcal{G}(x) x^{d/2}$.

Inserting this inverse-square asymptotic behaviour of $I(x')$ gives for the integral in (7.12) the scaling $\alpha_d \beta_d \gamma_d \lambda_d x^{-2}$ with

$$\beta_d = \int_0^1 \mathrm{d}y (1-y)^{(d-6)/2}(y^{(2-d)/2} - y^{(d+2)/2}) = -\frac{\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d)}. \tag{7.14}$$

It then follows, finally, that $C_E(t, t_{\mathrm{w}}) = \left(\alpha_d \beta_d \gamma_d \lambda_d^2/8\right) t_{\mathrm{w}}^{d/2} t^{-2}$ for $t \gg t_{\mathrm{w}}$ and

$$C_E'(t, t_{\mathrm{w}}) = \left(\alpha_d \beta_d \gamma_d \lambda_d^2 d/16\right) t_{\mathrm{w}}^{(d-2)/2} t^{-2}. \tag{7.15}$$

To get the FDR, we now need the response function R_E. This can be evaluated very similarly to the case $d > 4$ and one finds that the last term in (6.19) is again the dominant one, giving

$$R_E(t, t_{\mathrm{w}}) = \tfrac{1}{4} L_{\mathrm{eq}}^{(2)}(t - t_{\mathrm{w}}) \mathcal{F}_L(t/t_{\mathrm{w}}) = \tfrac{1}{4} L^{(2)}(t, t_{\mathrm{w}}). \tag{7.16}$$

(The $R\Omega C(t, t_{\mathrm{w}})$ terms in (6.19) look dangerous: they scale as $(t - t_{\mathrm{w}})^{-d/2}$ and are thus *larger* than $L_{\mathrm{eq}}^{(2)}(t - t_{\mathrm{w}}) \sim (t - t_{\mathrm{w}})^{(d-6)/2}$ in $d < 3$. However, their prefactors $-2T_{\mathrm{c}} + \hat{L}_{\mathrm{eq}}^{(2)}(0)$ cancel. Treating this cancellation more carefully then shows that these terms do remain subleading compared to (7.16).) As before, $4R_E$ equals $L^{(2)}$ in the ageing regime, and this then holds across the whole long-time regime since for $t - t_{\mathrm{w}} = \mathcal{O}(1)$ it matches the equilibrium behaviour $4R_E(t, t_{\mathrm{w}}) = L_{\mathrm{eq}}^{(2)}(t - t_{\mathrm{w}})$. For $t \gg t_{\mathrm{w}}$, on the other hand, the response (7.16) becomes

$$R_E(t, t_{\mathrm{w}}) = \frac{1}{4}\lambda_d t^{(d-6)/2}\left(\frac{t}{t_{\mathrm{w}}}\right)^{(2-d)/2} = \frac{\lambda_d}{4} t_{\mathrm{w}}^{(2-d)/2} t^{-2}. \tag{7.17}$$

Comparing with (7.15) then finally gives for the asymptotic FDR for the energy in $d < 4$,

$$X_E^{\infty} = \frac{4T_{\mathrm{c}}}{\alpha_d \beta_d \gamma_d \lambda_d d}. \tag{7.18}$$

After evaluating the various numerical factors (see appendix A), this can be written as

$$X_E^{\infty} = \frac{2}{\mathrm{d}\tilde{\alpha}_d} \frac{\Gamma(d)\Gamma\left(\frac{4-d}{2}\right)}{\Gamma\left(\frac{d+4}{2}\right)} \tag{7.19}$$

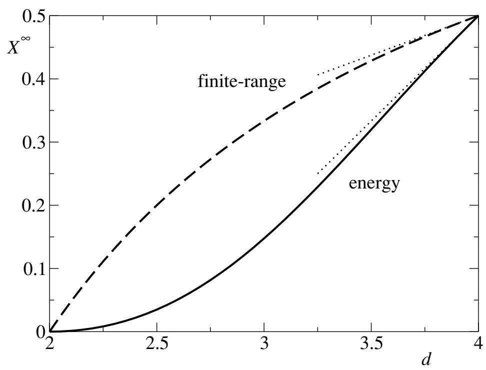

Figure 3. Asymptotic FDR X^∞ versus dimension d for finite-range observables (equation (3.3), dashed) and energy (equation (7.20), solid). Dotted lines indicate the corresponding linear expansions $1/2 - \epsilon/8$ and $1/2 - \epsilon/3$ in $\epsilon = 4 - d$.

where

$$\tilde{\alpha}_d = \frac{\pi}{\sin[\pi(4-d)/2]} + \int_1^\infty \mathrm{d}x \int_0^1 \mathrm{d}y\, y^{(d-4)/2} \frac{(x-y)^{(2-d)/2}}{1+x-y}\left(1 - y - x^{-(d+2)/2}\right). \qquad (7.20)$$

Near $d = 4$, one can expand in $\epsilon = 4 - d$. It can be shown by explicit calculation that the integral in (7.20) is exactly zero for $d = 4$, giving $\tilde{\alpha}_d = 2/\epsilon + \mathcal{O}(\epsilon)$ and so

$$X_E^\infty = \frac{\epsilon\Gamma(4-\epsilon)\Gamma(\epsilon/2)}{(4-\epsilon)\Gamma(4-\epsilon/2)} + \mathcal{O}(\epsilon^2) = \frac{1}{2} - \frac{\epsilon}{3} + \mathcal{O}(\epsilon^2). \qquad (7.21)$$

Notably, this is *different* from the FDR $X^\infty = 1 - 2/d = 1/2 - \epsilon/8 + \mathcal{O}(\epsilon^2)$ for all the other, finite-range, observables that we considered previously in $d < 4$. It is, however, consistent with RG calculations to $\mathcal{O}(\epsilon)$ for the $O(n)$-model in the limit $n \to \infty$, for an analogous choice of observable [13]. Remarkably, therefore, the non-Gaussian fluctuations induced by the weak infinite-range interaction in the spherical model seem to mimic precisely the effects that are seen in more realistic models such as $O(n)$, even though in the latter all interactions are short ranged and there is no difference between the behaviour of block observables and global ones.

In figure 3, we show the dependence of the asymptotic energy-FDR X_E^∞ on dimension d and compare with the result for finite-range observables. They agree in $d \geqslant 4$, but the difference between them grows as d decreases towards $d = 2$, with the energy FDR always having the lower value. In the limit $d \to 2$, both FDRs converge to zero, but while the finite-range FDR $X^\infty = \epsilon'/2 + \mathcal{O}(\epsilon'^2)$ does so linearly in $\epsilon' = d - 2$, the energy FDR vanishes quadratically as $X_E^\infty = \epsilon'^2/8 + \mathcal{O}(\epsilon'^3)$, due to the divergence of $\tilde{\alpha}_d = 4/\epsilon'^2 + \mathcal{O}(1/\epsilon')$.

As in the case $d > 4$, an energy FD plot would have a pseudo-equilibrium form which hides all non-equilibrium effects at long times. Indeed, one could write (7.12) in the form $C_E(t, t_\mathrm{w}) = C_E^{\mathrm{eq}}(t - t_\mathrm{w})\mathcal{F}_{C_E}(x)$, and the decay of the equilibrium factor C_E^{eq} squeezes all details about the ageing factor $\mathcal{F}_{C_E}(x)$ into a vanishingly small region of the FD plot for long times. While we have not calculated $\mathcal{F}_{C_E}$ explicitly, the discussion after (7.12) shows that $\mathcal{F}_{C_E}(1) = 1$ as it should be. For large x, on the other hand, we noted in (7.15) that $C_E(t, t_\mathrm{w}) \sim t_\mathrm{w}^{d/2} t^{-2}$, implying $\mathcal{F}_{C_E}(x) \sim t_\mathrm{w}^{d/2} t^{-2}(t - t_\mathrm{w})^{(4-d)/2} \sim x^{-d/2}$. This matches continuously at $d = 4$ with the prediction (6.23) for $d > 4$, where the ageing correction decays as $\mathcal{F}_{CC}(x) \sim x^{-2}$.

8. Magnetized initial states

In this final section, we consider the dynamics for initial configurations with nonzero magnetization, focusing as before on the non-equilibrium dynamics that results when the system is subsequently brought to the critical temperature T_c. Physically, such a situation could arise in an 'up-quench', where the system is equilibrated at $T < T_c$ and temperature is then increased to $T = T_c$. As explained in the introduction, our interest in this scenario arises from recent results [20] which show that such initial conditions produce FDRs that differ nontrivially from those for zero magnetization. The analysis of [20] was limited to high dimensions d or infinite-range interactions, however; the calculation below will allow us to see explicitly how the results change in finite dimension. In particular, we will obtain exact FDRs for magnetized coarsening below the upper critical dimension, i.e. $d < 4$ in the spherical model.

We will continue to use the notation $C_{\mathbf{q}}(t, t_w) = (1/N)\langle S_{\mathbf{q}}(t) S^*_{\mathbf{q}}(t)\rangle$ for Fourier mode correlations. For $\mathbf{q} = \mathbf{0}$ this is now a full, unsubtracted correlator, with $C_{\mathbf{0}}(t, t) = Nm^2(t) + \mathcal{O}(1)$ and $m(t) = (1/N)\langle S_{\mathbf{0}}(t)\rangle$ the time-dependent magnetization. The difference $\tilde{C}_{\mathbf{0}}(t, t_w) = C_{\mathbf{0}}(t, t_w) - Nm(t)m(t_w)$ is the connected correlator, which has values of $\mathcal{O}(1)$ and is the relevant quantity for analysing the FD behaviour. For $\mathbf{q} \neq \mathbf{0}$, connected and full correlators coincide:

$$\tilde{C}_{\mathbf{q}}(t, t_w) = C_{\mathbf{q}}(t, t_w) - (1/N)\langle S_{\mathbf{q}}(t)\rangle\langle S^*_{\mathbf{q}}(t_w)\rangle = C_{\mathbf{q}}(t, t_w). \tag{8.1}$$

We now need to check how the analysis in the previous sections is modified by the presence of a nonzero magnetization. The Fourier space equation of motion (2.5) remains valid, and so do the resulting expressions for the response function $R_{\mathbf{q}}(t, t_w)$ (2.6) and the full correlator $C_{\mathbf{q}}(t, t_w)$ (2.10), (2.13). The expression (2.16) for the Laplace transform of $g(t)$ that results from the spherical constraint also still holds, but the solution is now different. In the $\mathbf{q}$-integral, the $\mathbf{q} = \mathbf{0}$ contribution $(1/N)C_{\mathbf{0}}(0, 0)/s = m^2(0)/s$ has to be treated separately. In fact, one sees that for $s \to 0$ this term always dominates the rest of the integral, which diverges less strongly. At criticality, where $T = T_c = \left[\int(\mathrm{d}q)1/\omega\right]^{-1} = [2\hat{f}(0)]^{-1}$, one thus has for $s \to 0$

$$2T_c\hat{g}(s) = \frac{m^2(0)}{s}\left[\int(\mathrm{d}q)\left(\frac{1}{2\omega} - \frac{1}{s + 2\omega}\right)\right]^{-1}, \tag{8.2}$$

which using (4.22) can be rearranged into

$$\frac{\hat{g}(s)}{m^2(0)} = s^{-2}\left[\int(\mathrm{d}q)\frac{T_c}{\omega(2\omega + s)}\right]^{-1} = [s^2\hat{K}_{\mathrm{eq}}(s)]^{-1}. \tag{8.3}$$

For $d > 4$, $\hat{K}_{\mathrm{eq}}(0)$ is finite so this scales as s^{-2}; for $d < 4$, on the other hand, $\hat{K}_{\mathrm{eq}}(s)$ diverges as $s^{(d-4)/2}$ so that $\hat{g}(s) \sim s^{-d/2}$. Translating back to the time domain, $g(t)$ behaves for large t as

$$g(t) \sim m^2(0)t^{\alpha}, \qquad \alpha = \begin{cases} 1 & (d > 4) \\ (d-2)/2 & (d < 4). \end{cases} \tag{8.4}$$

Note that this asymptotic behaviour is independent of any details of the initial condition except for the presence of a nonzero $m(0)$; it depends on the actual value of $m(0)$ only through the prefactor $m^2(0)$. For the time dependence of $m(t)$, one gets by taking an average of (2.8)

$$m(t) = R_{\mathbf{0}}(t, 0)m(0) = \frac{m(0)}{\sqrt{g(t)}}. \tag{8.5}$$

Because of the proportionality of $g(t)$ to $m^2(0)$ for large t, the asymptotic decay of $m(t)$ is independent of the initial conditions, in terms of *both* the decay exponent *and* the prefactor.

8.1. Finite-range spin observables

We first analyse the correlation and response functions for observables that relate to a number of spins that is much smaller than N. As for the unmagnetized case, the fluctuations of the Lagrange multiplier z can then be neglected. To understand the magnetized case, it is useful to shift the spin variables by $m(t)$. We will see that the equations of motion then acquire the same form as before, so that we can directly transfer the main results from the unmagnetized case. Explicitly, we consider the following decomposition of the spin variables:

$$S_i = m + U_i \tag{8.6}$$

where U_i is a zero-mean variable. The equation of motion (2.4) for S_i then gives

$$\partial_t m + \partial_t U_i = -\Omega_{ij}(m + U_j) + \xi_i - z(m + U_i). \tag{8.7}$$

From (8.5) and definition (2.7), $\partial_t \ln m(t) = -(1/2)\partial_t \ln g(t) = -z(t)$, so $\partial_t m = -zm$. Also $\sum_j \Omega_{ij} = 0$, giving

$$\partial_t U_i = -\Omega_{ij} U_j + \xi_i - zU_i. \tag{8.8}$$

This is the same as the equation for S_i in the unmagnetized case, and so one can directly deduce the solutions for the correlation functions of U_i; these are the connected correlations $\tilde{C}$. The initial values again become unimportant for long times, allowing us to work out the scaling of the $\tilde{C}$, and then together with the response R also the FDR X. It is clear from the description in terms of the subtracted spins U_i that there is nothing special about the case $\mathbf{q} = \mathbf{0}$, and all results will have a smooth limit as $\mathbf{q} \to \mathbf{0}$. Because we are neglecting the fluctuations of the Lagrange multiplier z, this limit again has to be understood as that of a block magnetization calculated over a region much larger than the correlation length (in the time regime being considered) but much smaller than the linear system size, so that $q \gg 1/L$.

Applying (2.10), we can now directly write the connected correlation function at equal times as

$$\tilde{C}_{\mathbf{q}}(t_w, t_w) = \tilde{C}_{\mathbf{q}}(0, 0)\frac{e^{-2\omega t_w}}{g(t_w)} + 2T_c \frac{e^{-2\omega t_w}}{g(t_w)} \int_0^{t_w} dt'\, e^{2\omega t'} g(t'). \tag{8.9}$$

At long times, the first term is subleading due to the scaling (8.4), and one has the behaviour

$$\tilde{C}_{\mathbf{q}}(t_w, t_w) = \frac{T_c}{\omega}\mathcal{F}_C(\omega t_w), \qquad \mathcal{F}_C(w) = 2w \int_0^1 dy\, y^{\alpha}\, e^{-2w(1-y)}. \tag{8.10}$$

This result is of course the same as (3.4), except for the replacement of $-\kappa$ by α which reflects the difference in the asymptotic behaviour of $g(t)$.

The two-time connected correlations are $\tilde{C}_{\mathbf{q}}(t, t_w) = R_{\mathbf{q}}(t, t_w)\tilde{C}_{\mathbf{q}}(t_w, t_w)$ with $R_{\mathbf{q}}(t, t_w)$ given by (2.6) as before. As a consequence, expression (3.1) for the FDR $X_{\mathbf{q}}$ also remains valid, and one finds the scaling form $X_{\mathbf{q}}(t, t_w) = \mathcal{F}_X(\omega t_w)$ with

$$\mathcal{F}_X^{-1}(w) = 2 - (2w + \alpha) \int_0^1 dy\, y^{\alpha}\, e^{-2w(1-y)} \tag{8.11}$$

which is directly analogous to (3.4). In the limit $w = \omega t_w \to \infty$, $X_{\mathbf{q}} = 1$, which means that all modes with $\mathbf{q} \neq 0$ equilibrate once $t_w \gg 1/\omega$. In the opposite limit $w \to 0$, corresponding to $t_w \ll 1/\omega$,

$$X_{\mathbf{q}} = \frac{\alpha + 1}{\alpha + 2} = \begin{cases} 2/3 & (d > 4) \\ d/(d + 2) & (d < 4). \end{cases} \tag{8.12}$$

For $\mathbf{q} \to \mathbf{0}$, this result applies independently of the value of t_w as long as $t_w \gg 1$, so that the FDR for the block magnetization will be a straight line with slope (8.12). This is as for the unmagnetized case, but the actual value of the FDR is now *different*. It is also different from the value $X^\infty = 4/5$ predicted for Ising models in the limit of large d [20]; we will see below that the latter value is obtained for the *global* magnetization, which is affected by local spin fluctuations of $\mathcal{O}(N^{-1/2})$.

For later reference, we write the long-time forms of the correlation and response functions for $\mathbf{q} \to \mathbf{0}$. By setting $\omega = 0$ in (8.9) and taking the long-time limit where the first term becomes negligible, we find for the connected equal-time correlator

$$\tilde{C}_{\mathbf{0}}(t_\mathrm{w}, t_\mathrm{w}) = \frac{2T_\mathrm{c}t_\mathrm{w}}{1+\alpha}. \tag{8.13}$$

The response function is, from (2.6) and (8.5),

$$R_{\mathbf{0}}(t, t_\mathrm{w}) = \sqrt{\frac{g(t_\mathrm{w})}{g(t)}} = \frac{m(t)}{m(t_\mathrm{w})} = \left(\frac{t_\mathrm{w}}{t}\right)^{\alpha/2} \tag{8.14}$$

where the last equality holds for long times. The two-time correlator is therefore

$$\tilde{C}_{\mathbf{0}}(t, t_\mathrm{w}) = \frac{2T_\mathrm{c}t_\mathrm{w}}{1+\alpha}\left(\frac{t_\mathrm{w}}{t}\right)^{\alpha/2}. \tag{8.15}$$

From these results one of course retrieves the long-time FDR $X_{\mathbf{0}}(t, t_\mathrm{w}) = T_\mathrm{c}R_{\mathbf{0}}(t, t_\mathrm{w})/\tilde{C}'_{\mathbf{0}}(t, t_\mathrm{w}) = (\alpha+1)/(\alpha+2)$, obtained in (8.12) via the limit $\omega t_\mathrm{w} \to 0$. As explained above, these results apply in the regime $1/L \ll q \ll 1$. For $\mathbf{q} = \mathbf{0}$ itself, they capture only the Gaussian part of the spin fluctuations, and non-Gaussian corrections become relevant as discussed in the next section.

8.2. General expressions for magnetization correlation and response

We now turn to the FD behaviour of the global magnetization, corresponding to $\mathbf{q} = \mathbf{0}$ rather than $q \gg 1/L$. All N spins are now involved and one needs to account for the fluctuating contribution of the Lagrange multiplier, which we write as $z + N^{-1/2}\Delta z$ as before. To understand why this is necessary in the magnetized case, but was not in the unmagnetized scenario, consider the equation of motion (2.5) for the zero-wavevector Fourier component of the spins,

$$\partial_t S_{\mathbf{0}} = -(z + N^{-1/2}\Delta z)S_{\mathbf{0}} + \xi_{\mathbf{0}}. \tag{8.16}$$

In the unmagnetized case, $S_{\mathbf{0}}$ is a zero-mean quantity of $\mathcal{O}(N^{1/2})$. The Δz-term then contributes only subleading $\mathcal{O}(1)$ fluctuations. For nonzero magnetization, on the other hand, the mean of $S_{\mathbf{0}}$ is Nm, with fluctuations around this of $\mathcal{O}(N^{1/2})$. The coupling of Δz with m then gives an $\mathcal{O}(N^{1/2})$ contribution to $\partial_t S_{\mathbf{0}}$, which is no longer negligible.

To find the resulting non-Gaussian fluctuations in $S_{\mathbf{0}}$ explicitly, we make the decomposition $S_i = s_i + N^{-1/2}r_i$ as before. The discussion in section 4 then goes through, and we retrieve (4.11) for the $\mathcal{O}(N^{-1/2})$-corrections r_i to the spins. For the zero Fourier mode, in particular, we have

$$r_{\mathbf{0}}(t) = -\frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t''\,R_{\mathbf{0}}(t, t')s_{\mathbf{0}}(t')L(t', t'')\Delta(t''). \tag{8.17}$$

To simplify the calculation of connected correlations, we now decompose the Gaussian part of the spins into $s_i = m + u_i$, so that the u_i are zero-mean Gaussian variables. This corresponds to a decomposition $U_i = u_i + \sqrt{N}r_i$ of the fluctuating parts of the spins into leading Gaussian terms and small non-Gaussian corrections, in analogy to the representation $S_i = s_i + \sqrt{N}r_i$ in the unmagnetized case. The u_i obey the equation of motion (8.8), and their correlation and response functions are $\tilde{C}_{\mathbf{q}}$ and $R_{\mathbf{q}}$ calculated previously.

We will write the connected correlation function for the global magnetization which *includes* non-Gaussian corrections as $C_{\mathrm{m}}(t,t_{\mathrm{w}}) = (1/N)\langle U_{\mathbf{0}}(t)U_{\mathbf{0}}(t_{\mathrm{w}})\rangle$. Making the decomposition into Gaussian and non-Gaussian parts, this reads

$$C_{\mathrm{m}}(t,t_{\mathrm{w}}) = \frac{1}{N}\langle[u_{\mathbf{0}}(t) + N^{-1/2}r_{\mathbf{0}}(t)][u_{\mathbf{0}}(t_{\mathrm{w}}) + N^{-1/2}r_{\mathbf{0}}(t_{\mathrm{w}})]\rangle \tag{8.18}$$

with, from (8.17),

$$r_{\mathbf{0}}(t) = -\frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t''\,R_{\mathbf{0}}(t,t')[Nm(t') + u_{\mathbf{0}}(t')]L(t',t'')\Delta(t'') \tag{8.19}$$

$$= -\frac{N}{2}\int \mathrm{d}t'\,M(t,t')\Delta(t'). \tag{8.20}$$

Here, we have defined

$$M(t,t_{\mathrm{w}}) = \int \mathrm{d}t'\,R_{\mathbf{0}}(t,t')m(t')L(t',t_{\mathrm{w}}) = m(t)\int^{t} \mathrm{d}t'\,L(t',t_{\mathrm{w}}) \tag{8.21}$$

where the second form follows from (8.14). M is, like R and L, causal and so vanishes for $t < t_{\mathrm{w}}$. In (8.20), we have also discarded the Gaussian fluctuation term $u_{\mathbf{0}}$, which is of $\mathcal{O}(N^{1/2})$ and so negligible against the term Nm. This is in line with the intuition discussed earlier that non-Gaussian fluctuations arise only from the coupling of Δz to m. Note also that $r_{\mathbf{0}}$ is $\mathcal{O}(N)$, so that in (8.18) the non-Gaussian correction $N^{-1/2}r_{\mathbf{0}}$ is of the same order as the Gaussian fluctuation $u_{\mathbf{0}}$, again as expected.

Substituting (8.20) into (8.18), we see that we need the two-time correlations of $u_{\mathbf{0}}$ and Δ. In the presence of a nonzero m, the latter becomes

$$\Delta = \frac{1}{\sqrt{N}}\sum_i \left(s_i^2 - 1\right) = \frac{1}{\sqrt{N}}\sum_i \left(m^2 - 1 + 2u_i m + u_i^2\right). \tag{8.22}$$

The required correlations are therefore $\langle u_{\mathbf{0}}(t)u_{\mathbf{0}}(t_{\mathrm{w}})\rangle = N\tilde{C}_{\mathbf{0}}(t,t_{\mathrm{w}})$ and

$$\langle u_{\mathbf{0}}(t)\Delta(t')\rangle = \frac{1}{\sqrt{N}}\sum_{ij}\left\langle u_i(t)\left[m^2(t') - 1 + 2u_j(t')m(t') + u_j^2(t')\right]\right\rangle \tag{8.23}$$

$$= \frac{2}{\sqrt{N}}m(t')\sum_{ij}\langle u_i(t)u_j(t')\rangle = 2m(t')\sqrt{N}\tilde{C}_{\mathbf{0}}(t,t'). \tag{8.24}$$

For the autocorrelation of Δ, we can exploit the fact that $\langle\Delta\rangle = 0$ to write (8.22) as $\Delta = N^{-1/2}\sum_i \left(2u_i m + u_i^2 - \langle u_i^2\rangle\right)$. This gives

$$\langle\Delta(t')\Delta(t'_{\mathrm{w}})\rangle = \frac{1}{N}\sum_{ik}\left\langle\left[2u_i(t')m(t') + u_i^2(t') - \langle u_i^2(t')\rangle\right][\cdots t' \to t'_{\mathrm{w}}\cdots]\right\rangle \tag{8.25}$$

$$= \frac{4}{N}m(t')m(t'_{\mathrm{w}})\sum_{ij}\langle u_i(t')u_j(t'_{\mathrm{w}})\rangle + \frac{2}{N}\sum_{ij}\langle u_i(t')u_j(t'_{\mathrm{w}})\rangle^2 \tag{8.26}$$

$$= 4m(t')m(t'_{\mathrm{w}})\tilde{C}_{\mathbf{0}}(t',t'_{\mathrm{w}}) + 2\int (\mathrm{d}q)\tilde{C}_{\mathbf{q}}^2(t',t'_{\mathrm{w}}) \tag{8.27}$$

where we have used Wick's theorem to simplify the fourth-order average $\left\langle\left(u_i^2 - \langle u_i^2\rangle\right)\left(u_j^2 - \langle u_j^2\rangle\right)\right\rangle = \langle u_i^2 u_j^2\rangle - \langle u_i^2\rangle\langle u_j^2\rangle = 2\langle u_i u_j\rangle^2$. Abbreviating the q-integral as $\tilde{C}\tilde{C}(t',t'_{\mathrm{w}})$, the full connected correlation function (8.18) can thus be written as

$$
\begin{aligned}
C_{\mathrm{m}}(t,t_{\mathrm{w}}) = {} & \tilde{C}_{\mathbf{0}}(t,t_{\mathrm{w}}) - \frac{1}{2\sqrt{N}}\int \mathrm{d}t'\, M(t,t')\langle u_{\mathbf{0}}(t_{\mathrm{w}})\Delta(t')\rangle \\
& - \frac{1}{2\sqrt{N}}\int \mathrm{d}t' M(t_{\mathrm{w}},t')\langle u_{\mathbf{0}}(t)\Delta(t')\rangle \\
& + \frac{1}{4}\int \mathrm{d}t'\,\mathrm{d}t'_{\mathrm{w}} M(t,t')M(t_{\mathrm{w}},t'_{\mathrm{w}})\langle\Delta(t'_{\mathrm{w}})\Delta(t')\rangle \qquad (8.28)\\
= {} & C^{(1)}_{\mathrm{m}}(t,t_{\mathrm{w}}) + C^{(2)}_{\mathrm{m}}(t,t_{\mathrm{w}}) \qquad (8.29)
\end{aligned}
$$

where

$$
\begin{aligned}
C^{(1)}_{\mathrm{m}}(t,t_{\mathrm{w}}) = {} & \tilde{C}_{\mathbf{0}}(t,t_{\mathrm{w}}) - \int \mathrm{d}t'[M(t,t')\tilde{C}_{\mathbf{0}}(t_{\mathrm{w}},t') + M(t_{\mathrm{w}},t')\tilde{C}_{\mathbf{0}}(t,t')]m(t') \\
& + \int \mathrm{d}t'\,\mathrm{d}t'_{\mathrm{w}} M(t,t')M(t_{\mathrm{w}},t'_{\mathrm{w}})m(t')m(t'_{\mathrm{w}})\tilde{C}_{\mathbf{0}}(t',t'_{\mathrm{w}}) \qquad (8.30)\\
= {} & \int \mathrm{d}t'\,\mathrm{d}t'_{\mathrm{w}}[\delta(t-t') - M(t,t')m(t')][\delta(t_{\mathrm{w}}-t'_{\mathrm{w}}) - M(t_{\mathrm{w}},t'_{\mathrm{w}})m(t'_{\mathrm{w}})]\tilde{C}_{\mathbf{0}}(t',t'_{\mathrm{w}}) \\
& \qquad (8.31)
\end{aligned}
$$

and

$$C^{(2)}_{\mathrm{m}}(t,t_{\mathrm{w}}) = \frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t'_{\mathrm{w}} M(t,t')M(t_{\mathrm{w}},t'_{\mathrm{w}})\tilde{C}\tilde{C}(t',t'_{\mathrm{w}}). \qquad (8.32)$$

Next, we derive an expression for the corresponding magnetization response function. To this purpose, we expand the spins for small fields as

$$S_i = s_i + hr_i \qquad (8.33)$$

where s_i are the unperturbed spins and we neglect the $\mathcal{O}(N^{-1/2})$ non-Gaussian corrections as irrelevant, as in the unmagnetized case. The Lagrange multiplier is similarly written as $z + h\Delta z$. By collecting the $\mathcal{O}(h)$ terms from the equation of motion for S_i, we then find by analogy with (5.21) that r_i obey

$$\partial_t r_i = -\Omega_{ij}r_j - zr_i - \Delta z s_i + \delta(t-t_{\mathrm{w}}). \qquad (8.34)$$

Here, the last term represents a field impulse at time t_{w}, uniform over all sites as is appropriate for the field conjugate to the global magnetization. Since $r_i(t) = 0$ before the field is applied, i.e. for $t < t_{\mathrm{w}}$, this impulse perturbation gives $r_i(t_{\mathrm{w}}^+) = 1$. Starting from this value, we can then integrate (8.34) forward in time to get

$$r_i(t) = \sum_j R_{ij}(t,t_{\mathrm{w}}) - \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t' R_{ij}(t,t')\Delta z(t')s_j(t'). \qquad (8.35)$$

The condition we need to impose in order to get Δz is that the spherical constraint $(1/N)\sum_i (s_i + hr_i)^2 = 1$ needs to be satisfied to linear order in h, giving the condition $(1/N)\sum_i \langle r_i s_i\rangle = 0$. Inserting (8.35) into this yields

$$R_{\mathbf{0}}(t,t_{\mathrm{w}})m(t) = \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t' K(t,t')\Delta z(t') \qquad (8.36)$$

where we have used the definition (4.8) of $K(t,t')$. Applying the inverse kernel L gives

$$\Delta z(t) = \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t' L(t,t')m(t')R_{\mathbf{0}}(t',t_{\mathrm{w}}). \qquad (8.37)$$

Note that this result vanishes when $m = 0$, consistent with the fact that we did not need to consider perturbations of z in our calculation of the magnetization response in the unmagnetized case. We can now write the magnetization response function, which we denote by $R_{\rm m}(t, t_{\rm w})$. It is given by $R_{\rm m} = (1/N)\sum_i \langle r_i \rangle$; inserting the result for Δz into (8.35), we explicitly get

$$R_{\rm m}(t, t_{\rm w}) = R_{\mathbf{0}}(t, t_{\rm w}) - \int {\rm d}t''\,{\rm d}t'\,R_{\mathbf{0}}(t, t'')m(t'')L(t'', t')m(t')R_{\mathbf{0}}(t', t_{\rm w}) \tag{8.38}$$

$$= \int {\rm d}t'[\delta(t - t') - M(t, t')m(t')]R_{\mathbf{0}}(t', t_{\rm w}). \tag{8.39}$$

This completes the derivation of the general expressions for the magnetization correlation and response. To make progress, we need to find the kernel $M(t, t')$. This requires $L(t, t')$, which is the inverse of $K(t, t') = \int({\rm d}q)R_{\mathbf{q}}(t, t')C_{\mathbf{q}}(t, t')$. As is clear from the discussion in section 4, the correlator occurring here is the *unsubtracted* one. Because $C_{\mathbf{0}}(t, t') = Nm(t)m(t')$ is $\mathcal{O}(N)$, the $\mathbf{q} = \mathbf{0}$ term needs to be treated separately in spite of its vanishing weight $1/N$. It makes a contribution $(1/N)R_{\mathbf{0}}(t, t')Nm(t)m(t') = m^2(t)\theta(t - t')$, where we have simplified using (8.14). We can thus write

$$K(t, t') = \tilde{K}(t, t') + m^2(t)\theta(t - t') \tag{8.40}$$

with the $\mathbf{q} \neq \mathbf{0}$ contribution

$$\tilde{K}(t, t') = \int({\rm d}q)R_{\mathbf{q}}(t, t')\tilde{C}_{\mathbf{q}}(t, t'). \tag{8.41}$$

We have switched to the connected correlator here; this makes no difference for $\mathbf{q} \neq 0$, but allows us to include $\mathbf{q} = \mathbf{0}$ in the integral because $\tilde{C}_{\mathbf{0}} = \mathcal{O}(1)$. To say more, we will need to distinguish between dimensions $d > 4$ and $d < 4$.

8.3. *Magnetization correlation and response: non-equilibrium, $d > 4$*

The scaling of the connected part $\tilde{K}$ of K can be analysed exactly as in the case of zero magnetization: it consists of the same equilibrium time dependence modulated by an ageing function, $\tilde{K}(t, t') = K_{\rm eq}(t - t')\mathcal{F}_{\tilde{K}}(t/t')$. The ageing part can be worked out as in (4.15) with the only difference arising from the changed asymptotic behaviour of $g(t) \sim t^{\alpha}$ rather than $t^{-\kappa}$. For $\mathcal{F}_{\tilde{K}}$, we can therefore use directly (4.16), with κ replaced by $-\alpha$:

$$\mathcal{F}_{\tilde{K}}(x) = \frac{d-2}{2}(x-1)^{(d-2)/2}x^{-\alpha}\int_0^1 {\rm d}y\; y^{\alpha}(x-y)^{-d/2}. \tag{8.42}$$

In $d > 4$, where $\alpha = 1$, the integral can be computed explicitly to give

$$\mathcal{F}_{\tilde{K}}(x) = 1 - \frac{d-2}{d-4}\left(\frac{x-1}{x}\right) + \frac{2}{d-4}\left(\frac{x-1}{x}\right)^{(d-2)/2}. \tag{8.43}$$

We will see below that the precise behaviour of this function does not affect the results. Briefly though, for $x - 1 \ll 1$ the second term on the rhs is leading so that $\mathcal{F}_{\tilde{K}}$ decreases linearly with $x - 1$, while for large x one finds by expanding in $1/x$ that $\mathcal{F}_{\tilde{K}} \approx [(d-2)/4]x^{-2}$.

To find the inverse kernel L, consider how $K(t, t')$ varies with t. The first part in (8.40) starts off close to unity and decays on $\mathcal{O}(1)$ time scales $t - t'$ as $K_{\rm eq}(t - t')$, with a modulation by the ageing factor $\mathcal{F}_{\tilde{K}}(t/t')$ once $t - t'$ becomes comparable to t'. The second part, on the other hand, is small initially but only decays on ageing time scales. Comparing $K_{\rm eq}(t - t') \sim (t - t')^{(2-d)/2}$ to $m^2(t) \sim 1/t$, this second term therefore eventually becomes dominant, for $t - t' \sim t^{2/(d-2)}$. This discussion suggests that also the inverse kernel L should be composed of two parts with distinct long-time behaviour. We therefore write

$$L = \tilde{L} + L_{\mathbf{0}} \tag{8.44}$$

where $\tilde{L}$ is the inverse of $\tilde{K}$ and $L_{\mathbf{0}}$ arises from the zero-wavevector part of K. The continuous part of L is then similarly decomposed as $L^{(2)} = \tilde{L}^{(2)} + L_{\mathbf{0}}^{(2)}$.

We proceed by writing the defining equations for $L^{(2)}$ and $\tilde{L}^{(2)}$. The full inverse L is defined by (4.9) and as before has singular parts which are related to the behaviour of $K(t, t')$ for $t \to t'$. One can show directly from the definition of K, and exactly as in the unmagnetized case, that

$$K(t'^{+}, t') = 1, \qquad \partial_{t'} K(t, t')|_{t\to t'^{+}} = 2T_{\mathrm{c}}. \tag{8.45}$$

The decomposition (4.12) of the inverse kernel L therefore also remains valid, and from (4.9) and (8.40) we get the following equation for its continuous part $L^{(2)}$:

$$\int \mathrm{d}t' [\tilde{K}(t, t') + m^2(t)] L^{(2)}(t', t_{\mathrm{w}}) = 2T_{\mathrm{c}} \tilde{K}(t, t_{\mathrm{w}}) + 2T_{\mathrm{c}} m^2(t) - \partial_{t_{\mathrm{w}}} \tilde{K}(t, t_{\mathrm{w}}). \tag{8.46}$$

This is the analogue of the relation (4.19) for the case $m = 0$. We can argue similarly for $\tilde{L}$, which is defined by

$$\int \mathrm{d}t' \tilde{K}(t, t') \tilde{L}(t', t_{\mathrm{w}}) = \delta(t - t_{\mathrm{w}}). \tag{8.47}$$

From (8.40) and (8.45),

$$\tilde{K}(t'^{+}, t') = 1 - m^2(t), \qquad \partial_{t'} \tilde{K}(t, t')|_{t\to t'^{+}} = 2T_{\mathrm{c}} \tag{8.48}$$

and this initial behaviour implies that $\tilde{L}$ can be decomposed as

$$\tilde{L}(t', t_{\mathrm{w}}) = \frac{\delta'(t' - t_{\mathrm{w}})}{1 - m^2(t_{\mathrm{w}})} + \frac{2T_{\mathrm{c}}}{[1 - m^2(t_{\mathrm{w}})]^2} \delta(t' - t_{\mathrm{w}}) - \tilde{L}^{(2)}(t', t_{\mathrm{w}}). \tag{8.49}$$

Inserting into definition (8.47) gives for the continuous part $\tilde{L}^{(2)}$

$$\int \mathrm{d}t' \tilde{K}(t, t') \tilde{L}^{(2)}(t', t_{\mathrm{w}}) = \frac{2T_{\mathrm{c}}}{[1 - m^2(t_{\mathrm{w}})]^2} \tilde{K}(t, t_{\mathrm{w}}) - \frac{1}{1 - m^2(t_{\mathrm{w}})} \partial_{t_{\mathrm{w}}} \tilde{K}(t, t_{\mathrm{w}}). \tag{8.50}$$

Now for long times, we can approximate $1 - m^2(t_{\mathrm{w}}) \approx 1$. Then, (8.50) becomes identical to the relation (4.19) which determined $L^{(2)}$ in the unmagnetized case. Since $\tilde{K}$ has the same scaling form as K in (4.19), except for the replacement of $\mathcal{F}_K$ by $\mathcal{F}_{\tilde{K}}$, the solution for $\tilde{L}^{(2)}$ can be found in exactly the same way. In particular, the scaling functions describing the ageing corrections in $\tilde{K}$ and $\tilde{L}^{(2)}$ are again identical, and we can directly write

$$\tilde{L}^{(2)}(t, t') = L_{\mathrm{eq}}^{(2)}(t - t') \mathcal{F}_{\tilde{K}}(t/t') \tag{8.51}$$

as the long-time form of $\tilde{L}^{(2)}$. Here, $L_{\mathrm{eq}}^{(2)}$ is the same function as in the unmagnetized case, with Laplace transform (4.21).

It now remains to find $L_{\mathbf{0}}^{(2)}$. Subtracting (8.50) from (8.46) gives

$$\begin{aligned} \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t' \tilde{K}(t, t') L_{\mathbf{0}}^{(2)}(t', t_{\mathrm{w}}) + m^2(t) \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t' L_{\mathbf{0}}^{(2)}(t', t_{\mathrm{w}}) &= 2T_{\mathrm{c}} \frac{-2m^2(t_{\mathrm{w}}) + m^4(t_{\mathrm{w}})}{[1 - m^2(t_{\mathrm{w}})]^2} \tilde{K}(t, t_{\mathrm{w}}) \\ &\quad + m^2(t) \left[2T_{\mathrm{c}} - \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t' \tilde{L}^{(2)}(t', t_{\mathrm{w}}) \right] + \frac{m^2(t_{\mathrm{w}})}{1 - m^2(t_{\mathrm{w}})} \partial_{t_{\mathrm{w}}} \tilde{K}(t, t_{\mathrm{w}}). \end{aligned} \tag{8.52}$$

To make progress we assume that, by analogy with the structure (8.40) of K, $L_{\mathbf{0}}^{(2)}$ varies only on ageing time scales; we will find this confirmed *a posteriori*. We can then concentrate on the ageing regime $t - t_w \sim t_w \gg 1$. In this regime, the integral $\int_{t_w}^{t} dt' \tilde{L}^{(2)}(t', t_w) = \int_{t_w}^{t} dt' L_{eq}^{(2)}(t' - t_w)\mathcal{F}_{\tilde{K}}(t'/t_w)$ on the rhs of (8.52) becomes to leading order $\int_{t_w}^{\infty} dt' L_{eq}^{(2)}(t' - t_w) = \hat{L}_{eq}^{(2)}(0)$; the ageing correction $\mathcal{F}_{\tilde{K}}$ is unimportant because the integral converges for $t' - t_w = \mathcal{O}(1)$, and the upper integration limit can be set to infinity for the same reason. From (4.21), $\hat{L}_{eq}^{(2)}(0) = 2T_c - 1/\hat{K}_{eq}(0)$, and so the square bracket on the rhs of (8.52) becomes simply the constant $1/\hat{K}_{eq}(0)$. In the first and third terms on the rhs, on the other hand, $\tilde{K}$ and $\partial_{t_w}\tilde{K}$ scale as $(t - t_w)^{-(d-2)/2}$ and $(t - t_w)^{-d/2}$, respectively, and are negligible compared to the second term in the ageing regime. Disregarding these subleading terms, equation (8.52) is transformed to

$$\int_{t_w}^{t} dt' \tilde{K}(t, t') L_{\mathbf{0}}^{(2)}(t', t_w) + m^2(t) \int_{t_w}^{t} dt' L_{\mathbf{0}}^{(2)}(t', t_w) = \frac{m^2(t)}{\hat{K}_{eq}(0)}. \tag{8.53}$$

In the first integral, if as assumed $L_{\mathbf{0}}^{(2)}$ varies only on ageing time scales, the integral is dominated by the region $t' \approx t$ because of the factor $K_{eq}(t - t')$ in $\tilde{K}(t, t')$. This factor again makes the integral convergent within a region $t - t' = \mathcal{O}(1)$, and so we can approximate it by $\hat{K}_{eq}(0)L_{\mathbf{0}}^{(2)}(t, t_w)$. This gives

$$\int_{t_w}^{t} dt' L_{\mathbf{0}}^{(2)}(t', t_w) = \frac{1}{\hat{K}_{eq}(0)} - \frac{\hat{K}_{eq}(0)}{m^2(t)} L_{\mathbf{0}}^{(2)}(t, t_w) \tag{8.54}$$

and specifically in the limit $t/t_w \to 1$

$$L_{\mathbf{0}}^{(2)}(t_w, t_w) = \hat{K}_{eq}^{-2}(0) m^2(t_w). \tag{8.55}$$

To find $L_{\mathbf{0}}^{(2)}$ for $t/t_w > 1$, we use that $m^2(t) = \mu_d/t$ for large t and in $d > 4$, see (8.4), (8.5), with some dimension-dependent coefficient μ_d. Taking a derivative of (8.54) with respect to t then gives

$$L_{\mathbf{0}}^{(2)}(t, t_w)\big[1 + \mu_d^{-1}\hat{K}_{eq}(0)\big] = -t\mu_d{}^{-1}\hat{K}_{eq}(0)\partial_t L_{\mathbf{0}}^{(2)}(t, t_w). \tag{8.56}$$

This implies $\partial\big(\ln L_{\mathbf{0}}^{(2)}\big)\big/\partial(\ln t) = -[1 + \mu_d/\hat{K}_{eq}(0)]$ and so together with (8.55)

$$L_{\mathbf{0}}^{(2)}(t, t_w) = \hat{K}_{eq}^{-2}(0) \frac{\mu_d}{t_w} \left(\frac{t}{t_w}\right)^{-[1+\mu_d/\hat{K}_{eq}(0)]}. \tag{8.57}$$

This can be simplified because in fact $\mu_d = \hat{K}_{eq}(0)$. To see this, note from (8.3) that $\hat{g}(s)/m^2(0) = 1/[\hat{K}_{eq}(0)s^2]$ for small s; here, we have used that $\hat{K}_{eq}(0)$ is finite for $d > 4$. Transforming back to the time domain gives $g(t)/m^2(0) = m^{-2}(t) = t/\hat{K}_{eq}(0)$ for large t, i.e. $m^2(t) = \hat{K}_{eq}(0)/t$. Thus, (8.57) simplifies to

$$L_{\mathbf{0}}^{(2)}(t, t_w) = \frac{t_w}{\mu_d t^2}. \tag{8.58}$$

This result is consistent with our assumption that $L_{\mathbf{0}}^{(2)}$ only varies on ageing time scales. Overall, we have thus found for $L^{(2)}(t, t_w)$ the following long-time form:

$$L^{(2)}(t, t_w) = L_{eq}^{(2)}(t - t_w)\mathcal{F}_{\tilde{K}}(t/t_w) + \frac{t_w}{\mu_d t^2}. \tag{8.59}$$

Of course, for time differences $t - t_w = \mathcal{O}(1)$, $L_{\mathbf{0}}^{(2)}(t, t_w)$ will deviate from the form (8.58) derived for the ageing regime. However, one can verify by expanding both sides of (8.52) to $\mathcal{O}(t - t_w)$ that, even for $t = t_w$, $L_{\mathbf{0}}^{(2)}$ remains of order $1/t_w$, so that these small deviations are always negligible compared to the first term in (8.59).

We can now proceed to find M as defined in (8.21), and from there finally the explicit forms of the magnetization correlation and response functions. Using the general structure (4.12) of L, we have

$$m^{-1}(t)M(t,t_{\rm w}) = \delta(t-t_{\rm w}) + 2T_{\rm c} - \int_{t_{\rm w}}^{t} {\rm d}t'\, L^{(2)}(t',t_{\rm w}). \tag{8.60}$$

The integral can be separated into the contributions from the two parts of $L^{(2)}$ as given in (8.59). The first part yields an integral that converges for $t'-t_{\rm w} = \mathcal{O}(1)$; for $t - t_{\rm w} \gg 1$, it therefore gives $\hat{L}_{\rm eq}(0) = 2T_{\rm c} - 1/\hat{K}_{\rm eq}(0) = 2T_{\rm c} - 1/\mu_d$ to leading order. The second part, on the other hand, explicitly yields $\int_{t_{\rm w}}^{t} {\rm d}t'(t_{\rm w}/\mu_d t'^2) = \mu_d^{-1}(1-t_{\rm w}/t)$, so that

$$m^{-1}(t)M(t,t_{\rm w}) = \delta(t-t_{\rm w}) + \frac{1}{\mu_d} - \frac{1}{\mu_d}\left(1-\frac{t_{\rm w}}{t}\right) = \delta(t-t_{\rm w}) + \frac{1}{\mu_d}\frac{t_{\rm w}}{t}. \tag{8.61}$$

This result applies for $t - t_{\rm w} \gg 1$. For $t - t_{\rm w} = \mathcal{O}(1)$, it is not accurate; e.g., at $t = t_{\rm w}$ the continuous part of $m^{-1}(t)M(t,t_{\rm w})$ is, from (8.60), $2T_{\rm c}$ rather than $1/\mu_d$. However, this deviation over an $\mathcal{O}(1)$ time range only gives subleading corrections in the integrals over M that we need below, as indeed does the $\delta(t-t_{\rm w})$-term. This can be seen in (8.31) and (8.39), where only the combination $\delta(t-t_{\rm w}) - M(t,t_{\rm w})m(t_{\rm w})$ occurs; the latter can be written as

$$\delta(t-t_{\rm w}) - M(t,t_{\rm w})m(t_{\rm w}) = \delta(t-t_{\rm w})[1-m(t)m(t_{\rm w})] - \frac{t_{\rm w}^{1/2}}{t^{3/2}} = \delta(t-t_{\rm w}) - \frac{t_{\rm w}^{1/2}}{t^{3/2}}. \tag{8.62}$$

The second form holds for long times, where the $m(t)m(t_{\rm w})$ term that originated from the δ-term in (8.61) is negligible.

We can now work out expression (8.29) for the full connected correlation function. One can show that the contribution $C_{\rm m}^{(2)}$ involving $\tilde{C}\tilde{C}$ is negligible in the long-time limit. In expression (8.30) for the remainder $C_{\rm m}^{(1)}$, let us call the second and third terms I_2 and I_3. We need $\tilde{C}_{\mathbf{0}}$, which from (8.15) reads $\tilde{C}_{\mathbf{0}}(t',t'_{\rm w}) = T_{\rm c} t_{\rm w}'^{3/2} t'^{-1/2}$ for $t' > t'_{\rm w}$; because $\tilde{C}_{\mathbf{0}}$ is symmetric in time this then implies $\tilde{C}_{\mathbf{0}}(t',t'_{\rm w}) = T_{\rm c} t'^{3/2} t_{\rm w}'^{-1/2}$ for $t' < t'_{\rm w}$. Paying due attention to this temporal ordering of the arguments of $\tilde{C}_{\mathbf{0}}$ and using (8.62), one finds

$$I_2(t,t_{\rm w}) = T_{\rm c}\left[\int_0^{t_{\rm w}} {\rm d}t' \frac{t'^{1/2}}{t^{3/2}}\frac{t'^{3/2}}{t_{\rm w}^{1/2}} + \int_{t_{\rm w}}^{t} {\rm d}t' \frac{t'^{1/2}}{t^{3/2}}\frac{t_{\rm w}^{3/2}}{t'^{1/2}}\right] + T_{\rm c}\int_0^{t_{\rm w}} {\rm d}t' \frac{t'^{1/2}}{t_{\rm w}^{3/2}}\frac{t'^{3/2}}{t^{1/2}} \tag{8.63}$$

$$= T_{\rm c}\left(\frac{t_{\rm w}}{t}\right)^{3/2}\left(\frac{4}{3}t - \frac{2}{3}t_{\rm w}\right). \tag{8.64}$$

Similarly, the double integral in (8.30) can be evaluated as

$$I_3(t,t_{\rm w}) = \int_0^t {\rm d}t' \int_0^{t_{\rm w}} {\rm d}t'_{\rm w} \frac{t'^{1/2}}{t^{3/2}}\frac{t_{\rm w}'^{1/2}}{t_{\rm w}^{3/2}} \tilde{C}_{\mathbf{0}}(t'_{\rm w},t') \tag{8.65}$$

$$= T_{\rm c}\int_{t_{\rm w}}^{t} {\rm d}t' \int_0^{t_{\rm w}} {\rm d}t'_{\rm w} \frac{t'^{1/2}}{t^{3/2}}\frac{t_{\rm w}'^{1/2}}{t_{\rm w}^{3/2}}\frac{t_{\rm w}'^{3/2}}{t'^{1/2}} + 2T_{\rm c}\int_0^{t_{\rm w}} {\rm d}t' \int_0^{t'} {\rm d}t'_{\rm w} \frac{t'^{1/2}}{t^{3/2}}\frac{t_{\rm w}'^{1/2}}{t_{\rm w}^{3/2}}\frac{t_{\rm w}'^{3/2}}{t'^{1/2}} \tag{8.66}$$

$$= \frac{T_{\rm c}}{3}\left(\frac{t_{\rm w}}{t}\right)^{3/2}(t-t_{\rm w}) + \frac{2T_{\rm c}}{3}\int_0^{t_{\rm w}} {\rm d}t' \frac{t'^3}{t^{3/2}t_{\rm w}^{3/2}} \tag{8.67}$$

$$= \frac{T_{\rm c}}{6}\left(\frac{t_{\rm w}}{t}\right)^{3/2}(2t-t_{\rm w}). \tag{8.68}$$

Our final long-time result for the connected magnetization correlator including non-Gaussian corrections is then

$$C_{\rm m}(t,t_{\rm w}) = T_{\rm c}\frac{t_{\rm w}^{3/2}}{t^{1/2}} + I_2(t,t_{\rm w}) + I_3(t,t_{\rm w}) \tag{8.69}$$

$$= T_{\rm c}\left(\frac{t_{\rm w}}{t}\right)^{3/2}\left(t - \frac{4}{3}t + \frac{2}{3}t_{\rm w} + \frac{1}{3}t - \frac{1}{6}t_{\rm w}\right) = \frac{T_{\rm c}t_{\rm w}}{2}\left(\frac{t_{\rm w}}{t}\right)^{3/2}. \tag{8.70}$$

For the conjugate magnetization response function we get from (8.39) and (8.61), together with $R_{\mathbf{0}}(t,t_{\rm w}) = (t_{\rm w}/t)^{1/2}$,

$$R_{\rm m}(t,t_{\rm w}) = R_{\mathbf{0}}(t,t_{\rm w}) - \int_{t_{\rm w}}^{t} {\rm d}t'\,\frac{t'^{1/2}}{t^{3/2}}R_{\mathbf{0}}(t',t_{\rm w}) \tag{8.71}$$

$$= \left(\frac{t_{\rm w}}{t}\right)^{1/2} - \frac{t_{\rm w}^{1/2}}{t^{3/2}}(t-t_{\rm w}) = \left(\frac{t_{\rm w}}{t}\right)^{3/2}. \tag{8.72}$$

The FDR follows finally as

$$X_{\rm m}(t,t_{\rm w}) = \frac{T_{\rm c}R_{\rm m}(t,t_{\rm w})}{C'_{\rm m}(t,t_{\rm w})} = \frac{4}{5}. \tag{8.73}$$

Interestingly, this exactly agrees with the result for the Ising ferromagnet in the limit of large dimensionality d [20]. As in the unmagnetized case, we see therefore that it is the *global* observables in the spherical model, which are strongly affected by non-Gaussian fluctuations, that behave like their analogues in short-range models. In fact, even the expressions for the correlation and response functions we find here are identical to those in the large-d Ising case, implying 'universality' at a more detailed level than one might have expected.

It is worth noting that the effect of the non-Gaussian corrections is very large: compared to the Gaussian result $\tilde{C}_{\mathbf{0}}(t,t_{\rm w}) \sim t_{\rm w}(t_{\rm w}/t)^{1/2}$, the corrections increase the decay exponent to $C_{\rm m}(t,t_{\rm w}) \sim t_{\rm w}(t_{\rm w}/t)^{3/2}$, so that $C_{\rm m}/\tilde{C}_{\mathbf{0}} \sim t_{\rm w}/t \ll 1$ for $t \gg t_{\rm w}$: there is an almost perfect cancellation of Gaussian terms and non-Gaussian corrections for well-separated times. Similar comments apply to the response. The overall effect of the non-Gaussian corrections on the FD relation is to leave this as a straight line (since X is constant), but to increase the slope from $2/3$ to $4/5$.

8.4. Magnetization correlation and response: non-equilibrium, $d < 4$

We now consider systems below the upper critical dimension, $d < 4$; here there are no predictions yet from other models for the non-equilibrium FD behaviour following a quench of a magnetized initial state to $T_{\rm c}$ (but see section 9). As in the case of the energy correlations for unmagnetized initial states, leading-order cancellation effects have to be taken care of for these low values of d.

We again need to know the scaling of K to determine $L^{(2)}$; from this, we then get M and finally the correlation and response functions. The long-time scaling of the connected part of K is $\tilde{K}(t,t') = K_{\rm eq}(t-t')\mathcal{F}_{\tilde{K}}(t/t')$ as before, with $\mathcal{F}_{\tilde{K}}$ given by (8.42) but now $\alpha = (d-2)/2$. The contribution to K from $\mathbf{q} = \mathbf{0}$, given by second term in (8.40), is negligible relative to $\tilde{K}(t,t')$ for $t - t' = \mathcal{O}(1)$. However, for $t - t' \gg 1$, it becomes comparable and has the same overall time scaling as $\tilde{K}(t,t')$ in the ageing regime. To see this explicitly, recall from (8.4), (8.5) that the square of the magnetization decays asymptotically as $m^2(t) = \mu_d t^{-\alpha} = \mu_d t^{(2-d)/2}$ with some constant μ_d. Similarly, the equilibrium part of $\tilde{K}$

behaves as $K_{\mathrm{eq}}(t-t') = k_d(t-t')^{(2-d)/2}$ for $t - t' \gg 1$ (see after (4.13)). This gives

$$\frac{m^2(t)}{K_{\mathrm{eq}}(t-t')} = \frac{\mu_d(t-t')^{(d-2)/2}}{k_d t^{(d-2)/2}} = \frac{\mu_d}{k_d}\left(\frac{t/t'-1}{t/t'}\right)^{(d-2)/2} \tag{8.74}$$

in the ageing regime where both t' and $t-t'$ are large. (For $t-t' = \mathcal{O}(1)$, this expression is not accurate but this is irrelevant because there the term $m^2(t)$ is subleading compared to $\tilde{K}(t,t')$ anyway.) We thus have the overall scaling of K

$$K(t,t') = K_{\mathrm{eq}}(t-t')\mathcal{F}_K(t/t'), \qquad \mathcal{F}_K(x) = \mathcal{F}_{\tilde{K}}(x) + \frac{\mu_d}{k_d}\left(\frac{x-1}{x}\right)^{(d-2)/2}. \tag{8.75}$$

To simplify the scaling function, we integrate expression (8.42) by parts and rescale $y \to xy$, bearing in mind that $\alpha = (d-2)/2$:

$$\mathcal{F}_{\tilde{K}}(x) = \frac{d-2}{2}\left(\frac{x-1}{x}\right)^{(d-2)/2}\left[\frac{2}{d-2}(x-1)^{(2-d)/2} - \int_0^{1/x} \mathrm{d}y\, y^{(d-4)/2}(1-y)^{(2-d)/2}\right]. \tag{8.76}$$

For $x \to 1$, the integral becomes a beta function which evaluates to $\Gamma((d-2)/2)\Gamma((4-d)/2)$, and extracting this term gives

$$\begin{aligned}\mathcal{F}_{\tilde{K}}(x) = x^{(2-d)/2} - \Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{4-d}{2}\right)\left(\frac{x-1}{x}\right)^{(d-2)/2} \\ + \frac{d-2}{2}\left(\frac{x-1}{x}\right)^{(d-2)/2}\int_{1/x}^{1}\mathrm{d}y\, y^{(d-4)/2}(1-y)^{(2-d)/2}.\end{aligned} \tag{8.77}$$

The second term has the same dependence on x as the additional contribution from zero wavevector in (8.75). In fact, it turns out that these two terms cancel exactly: from our definitions of k_d and μ_d we have $K_{\mathrm{eq}}(t) = k_d t^{(2-d)/2}$ for large t, while $m^{-2}(t) = g(t)/m^2(0) = \mu_d^{-1}t^{(d-2)/2}$. Laplace transforming gives $\hat{K}_{\mathrm{eq}}(s) = k_d\Gamma((4-d)/2)s^{(d-4)/2}$ and $\hat{g}(s)/m^2(0) = \mu_d^{-1}\Gamma(d/2)s^{-d/2}$ to leading order for small s. But then (8.3) shows that $\mu_d^{-1}\Gamma(d/2) = [k_d\Gamma((4-d)/2)]^{-1}$ or $\mu_d/k_d = \Gamma(d/2)\Gamma((4-d)/2)$, proving the cancellation anticipated above. Overall, we thus have for the scaling function of K

$$\mathcal{F}_K(x) = x^{(2-d)/2} + \frac{d-2}{2}\left(\frac{x-1}{x}\right)^{(d-2)/2}\int_{1/x}^{1}\mathrm{d}y\, y^{(d-4)/2}(1-y)^{(2-d)/2}. \tag{8.78}$$

Expanding for $x \approx 1$, one sees that the leading-order variation is linear in $x-1$:

$$\begin{aligned}\mathcal{F}_K(x) &\approx 1 + \frac{2-d}{2}(x-1) + \frac{d-2}{2}\left(\frac{x-1}{x}\right)^{(d-2)/2}\int_{1/x}^{1}\mathrm{d}y(1-y)^{(2-d)/2} \\ &\approx 1 + \frac{1}{2}\frac{(d-2)^2}{4-d}(x-1).\end{aligned} \tag{8.79}$$

Note that the prefactor is positive, so that $\mathcal{F}_K(x)$ *increases* with x in the current scenario. This trend persists for all x, not just $x \approx 1$, and the scaling function monotonically approaches a limit value for $x \to \infty$. The latter follows from (8.75) as $\mu_d/k_d = \Gamma(d/2)\Gamma((4-d)/2)$, since the connected contribution $\mathcal{F}_{\tilde{K}}(x)$ decays to zero for $x \to \infty$.

With the single overall scaling (8.75) of K, we no longer need to decompose $L^{(2)}$ into $\tilde{L}^{(2)}$ and $L_{\mathbf{0}}^{(2)}$ as we did in $d > 4$; instead the long-time behaviour of $L^{(2)}$ will have the same structure as in the unmagnetized case,

$$L^{(2)}(t,t_{\mathrm{w}}) = L_{\mathrm{eq}}^{(2)}(t-t_{\mathrm{w}})\mathcal{F}_L(t/t_{\mathrm{w}}). \tag{8.80}$$

One can then exactly follow the discussion in section 4.2 to arrive at the integral equation (4.34) for $\mathcal{F}_L(x)$. Solving the latter looks a rather formidable task, given that

$\mathcal{F}_K$ itself has the relatively complicated form (8.78). Remarkably, however, the solution can be found in closed form and is given simply by

$$\mathcal{F}_L(x) = \frac{2}{4-d}x^{(2-d)/2} + \frac{2-d}{4-d}x^{-d/2}. \tag{8.81}$$

We were led to this result initially by a systematic series expansion of both $\mathcal{F}_K(x)$ and $\mathcal{F}_L(x)$ in terms of $(x-1)/x$. We do not detail this here, but verify in appendix B by direct calculation that (8.81) does indeed solve (4.34).

We next calculate the kernel $M(t, t_w)$. One inserts the scaling (8.80) into (8.60), subtracting off and adding back on the contribution from the equilibrium part of $L^{(2)}$:

$$m^{-1}(t)M(t, t_w) = \delta(t-t_w) + 2T_c - \int_{t_w}^{t} dt' L_{eq}^{(2)}(t'-t_w) + \int_{t_w}^{t} dt' L_{eq}^{(2)}(t'-t_w)\left[1 - \mathcal{F}_L\left(\frac{t'}{t_w}\right)\right]. \tag{8.82}$$

This is done to account for the leading-order cancellation of the second and third terms:

$$2T_c - \int_{t_w}^{t} dt' L_{eq}^{(2)}(t'-t_w) = 2T_c - \hat{L}_{eq}^{(2)}(0) + \int_{t}^{\infty} dt' L_{eq}^{(2)}(t'-t_w) = \frac{2\lambda_d}{4-d}(t-t_w)^{(d-4)/2} \tag{8.83}$$

where in the last step we have restricted ourselves to the ageing regime $t - t_w \gg 1$ and used the asymptotic behaviour (4.23), $L_{eq}^{(2)}(t'-t_w) = \lambda_d(t'-t_w)^{(d-6)/2}$. In the remaining integral in (8.82), the factor $1 - \mathcal{F}_L(t'/t_w) \sim (t'-t_w)/t_w$ ensures that the integration no longer has its weight concentrated near $t' = t_w$; put differently, after scaling the integration variable by t_w to $y = t'/t_w$ the integral is convergent at the lower end. We thus obtain in the ageing regime

$$m^{-1}(t)M(t, t_w) = \delta(t-t_w) + \lambda_d t_w^{(d-4)/2} \times \left\{\frac{2}{4-d}(x-1)^{(d-4)/2} + \int_1^x dy (y-1)^{(d-6)/2}[1 - \mathcal{F}_L(y)]\right\}. \tag{8.84}$$

Inserting (8.81) for $\mathcal{F}_L$, the integral in (8.84) can be done explicitly to give $[2/(4-d)]x^{(2-d)/2}(x-1)^{(d-4)/2}$ for the sum of the terms in curly brackets. A little care is needed here because the separate integrals over $(y-1)^{(d-6)/2}$ and $(y-1)^{(d-6)/2}\mathcal{F}_L(y)$ are divergent at the lower end. One can avoid this by analytical continuation from $d > 4$, where these divergences are absent or by integrating from $1+\epsilon$ to x and taking $\epsilon \to 0$ at the end. The δ-term is again subleading for long times in the relevant combination (8.62), which we can write as

$$\delta(t-t_w) - M(t, t_w)m(t_w) = \delta(t-t_w) - \frac{2}{4-d}\frac{\lambda_d \mu_d t_w^{(d-4)/2}}{t_w^{(d-2)/4} t^{(d-2)/4}} x^{(2-d)/2}(x-1)^{(d-4)/2}$$
$$= \delta(t-t_w) - \frac{1}{t}\mathcal{F}_M(x) \tag{8.85}$$

with

$$\mathcal{F}_M(x) = \frac{d-2}{2}x^{(2-d)/4}\left(\frac{x-1}{x}\right)^{(d-4)/2}. \tag{8.86}$$

Here, we have eliminated the constants λ_d and μ_d using the following argument. From (8.3), $\hat{g}(s)/m^2(0) = \hat{L}_{eq}(s)/s^2$ for small s, i.e. $\hat{L}_{eq}(s) = s^2\hat{g}(s)/m^2(0)$. In the time domain this gives at long times $L_{eq}^{(2)}(t) = -L_{eq}(t) = -(\partial_t)^2 m^{-2}(t) = -\mu_d^{-1}(\partial_t)^2 t^{(d-2)/2} = -\mu_d^{-1}[(d-2)/2][(d-4)/2]t^{(d-6)/2}$, so that $\lambda_d\mu_d = (d-2)(4-d)/4$.

Reassuringly, for $d \to 4$ the result (8.86) tends to $\mathcal{F}_M(x) = x^{-1/2}$, matching smoothly to the result (8.62) we found earlier in $d > 4$. For $d < 4$, the scaling function diverges as $x \to 1$. Since from (8.60) the continuous part of $M(t, t_w)$ is exactly given by $2T_c m(t)$ for equal times, this indicates that the above ageing regime expression must break down eventually when $t - t_w$ becomes small, as expected. In the integrals where M appears below such effects can be neglected, however, because they only give subleading corrections.

With the scaling of $\mathcal{F}_M$ in hand, we can now compute the connected correlation function $C_m(t, t_w)$ including non-Gaussian corrections, given by (8.29). After rescaling the integration variables, the first part (8.31) can be written as

$$C_m^{(1)}(t, t_w) = \int_0^x dy[\delta(y - x) - x^{-1}\mathcal{F}_M(x/y)] \int_0^1 dy_w[\delta(y_w - 1) - \mathcal{F}_M(1/y_w)]\tilde{C}_{\mathbf{0}}(t_w y, t_w y') \tag{8.87}$$

where the Gaussian magnetization correlator is

$$\tilde{C}_{\mathbf{0}}(t_w y, t_w y') = (4T_c t_w/d) \min\{y(y/y_w)^{(d-2)/4}, y_w(y_w/y)^{(d-2)/4}\} \tag{8.88}$$

from (8.15). At first sight, (8.87) suggests, e.g. from the $\delta(y - x)$ term, an asymptotic decay of $C_m^{(1)} \sim t_w x^{(2-d)/4}$ for large x. But this would not match continuously with the result (8.70) we found for $d > 4$. A cancellation of such leading-order terms must therefore occur for $x \to \infty$. To show this explicitly, we verify from (8.86) the identity

$$\int_0^x dy[\delta(y - x) - x^{-1}\mathcal{F}_M(x/y)]y^{-(d-2)/4}$$

$$= x^{(2-d)/4} - \frac{d-2}{2x}\int_0^x dy(x/y)^{(2-d)/4}(1 - y/x)^{(d-4)/2}y^{(2-d)/4} \tag{8.89}$$

$$= x^{(2-d)/4} - \frac{d-2}{2}x^{(2-d)/4}\int_0^1 dz(1 - z)^{(d-4)/2} = 0. \tag{8.90}$$

Multiplying this by $(4T_c t_w/d)\int_0^1 dy_w[\delta(y_w - 1) - \mathcal{F}_M(1/y_w)]y_w^{(d+2)/4}$ and subtracting from (8.87) exactly cancels all contributions in the range $y > y_w$, giving

$$C_m^{(1)}(t, t_w) = \frac{4T_c t_w}{dx}\int_0^1 dy_w \int_0^{y_w} dy \mathcal{F}_M(x/y)[\delta(y_w - 1) - \mathcal{F}_M(1/y_w)]$$
$$\times \left[y_w\left(\frac{y_w}{y}\right)^{(d-2)/4} - y\left(\frac{y}{y_w}\right)^{(d-2)/4}\right]. \tag{8.91}$$

This shows that $C_m^{(1)}/t_w$ is a scaling function of $x = t/t_w$. Its full x-dependence has to be found numerically from (8.91) or via series expansion [22], but we can obtain the large-x behaviour that is required for the asymptotic FDR X^∞ in closed form. For $x \to \infty$, one can replace the function $\mathcal{F}_M(x/y)$ with its asymptotic form $[(d-2)/2](x/y)^{(2-d)/4}$ from (8.86) to get

$$\frac{C_m^{(1)}(t, t_w)}{T_c t_w} = \frac{2(d-2)}{d}x^{-(d+2)/4}\int_0^1 dy_w \int_0^{y_w} dy\, y^{(d-2)/4}\left[\delta(y_w - 1) - \mathcal{F}_M\left(\frac{1}{y_w}\right)\right]$$
$$\times \left[y_w\left(\frac{y_w}{y}\right)^{(d-2)/4} - y\left(\frac{y}{y_w}\right)^{(d-2)/4}\right] \tag{8.92}$$

$$= \frac{2(d-2)}{d+2}x^{-(d+2)/4}\left[1 - \frac{d-2}{2}\frac{\Gamma\left(\frac{d+4}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(d+1)}\right]. \tag{8.93}$$

This exhibits the expected leading-order cancellation for large x, which gives an additional factor of $1/x$ compared to the naive result $x^{(2-d)/4}$.

To complete the calculation of the correlation function, we need to evaluate $C_{\mathrm{m}}^{(2)}$ from (8.32), which cannot be neglected for $d < 4$. This requires the long-time behaviour of $\tilde{C}\tilde{C}(t, t_{\mathrm{w}})$, which is given by (7.5) and (7.6) for $t > t_{\mathrm{w}}$ and $t < t_{\mathrm{w}}$, respectively, as for the unmagnetized case. The only modification arises from the different behaviour of $g(t)$. One thus finds

$$\frac{\tilde{C}\tilde{C}(t,t_{\mathrm{w}})}{\tilde{C}\tilde{C}(t,t)} = \mathcal{G}(t/t_{\mathrm{w}}), \qquad \mathcal{G}(x) = \begin{cases} \dfrac{\int \mathrm{d}w w^{(d-6)/2}\mathcal{F}_C^2(w)\,\mathrm{e}^{-2(x-1)w}}{x\int \mathrm{d}w w^{(d-6)/2}\mathcal{F}_C^2(w)} & \text{for } x \geqslant 1 \\ x^{(d-6)/2}\mathcal{G}(1/x) & \text{for } x \leqslant 1. \end{cases} \tag{8.94}$$

The scaling of the equal-time value of $\tilde{C}\tilde{C}$ is, from (7.5), $\tilde{C}\tilde{C}(t,t) = \tilde{\gamma}_d t^{(4-d)/2}$ with $\tilde{\gamma}_d = \sigma_d T_{\mathrm{c}}^2 \int \mathrm{d}w\, w^{(d-6)/2}\mathcal{F}_C^2(w)$; compare (A.1). Inserting this into (8.32) gives

$$C_{\mathrm{m}}^{(2)}(t,t_{\mathrm{w}}) = \frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t_{\mathrm{w}}' M(t,t')M(t_{\mathrm{w}},t_{\mathrm{w}}')\tilde{C}\tilde{C}(t',t_{\mathrm{w}}') \tag{8.95}$$

$$= \frac{1}{2}\int \mathrm{d}t'\,\mathrm{d}t_{\mathrm{w}}' \frac{1}{m(t')m(t_{\mathrm{w}}')}\frac{1}{t}\mathcal{F}_M(t/t')\frac{1}{t_{\mathrm{w}}}\mathcal{F}_M(t_{\mathrm{w}}/t_{\mathrm{w}}')\tilde{C}\tilde{C}(t',t')\mathcal{G}(t',t_{\mathrm{w}}') \tag{8.96}$$

$$= \frac{t_{\mathrm{w}}\tilde{\gamma}_d}{2\mu_d x}\int_0^1 \mathrm{d}y_{\mathrm{w}}\, y_{\mathrm{w}}^{(d-2)/4}\mathcal{F}_M(1/y_{\mathrm{w}})\int_0^x \mathrm{d}y\mathcal{F}_M(x/y)y^{(6-d)/4}\mathcal{G}(y/y_{\mathrm{w}}) \tag{8.97}$$

$$= \frac{t_{\mathrm{w}}}{x}\int_0^1 \mathrm{d}y_{\mathrm{w}}\, y_{\mathrm{w}}^2\,\mathcal{F}_M(1/y_{\mathrm{w}})\int_0^{x/y_{\mathrm{w}}} \mathrm{d}u\,\mathcal{F}_M(x/uy_{\mathrm{w}})u^{(6-d)/4}\frac{\tilde{\gamma}_d}{2\mu_d}\mathcal{G}(u). \tag{8.98}$$

In the second line we have used (8.85) to write $M(t,t') = m(t)\delta(t-t') - t^{-1}m^{-1}(t')\mathcal{F}_M(t/t')$ up to negligible corrections, and then immediately discarded the δ-function contributions, which give subleading corrections.

Let us denote by U the value of the u-integral in (8.98). Since $\mathcal{G}(u)$ is defined separately for $u > 1$ and $u < 1$ in (8.94), one splits the integral accordingly:

$$\frac{2\mu_d U}{\tilde{\gamma}_d} = \int_0^1 \mathrm{d}u\,\mathcal{F}_M(x/uy_{\mathrm{w}})u^{(d-6)/4}\mathcal{G}(1/u) + \int_1^{x/y_{\mathrm{w}}} \mathrm{d}u\,\mathcal{F}_M(x/uy_{\mathrm{w}})u^{(6-d)/4}\mathcal{G}(u) \tag{8.99}$$

$$= \int_1^{\infty} \mathrm{d}u\,\mathcal{F}_M(xu/y_{\mathrm{w}})u^{-(d+2)/4}\mathcal{G}(u) + \int_1^{x/y_{\mathrm{w}}} \mathrm{d}u\,\mathcal{F}_M(x/uy_{\mathrm{w}})u^{(6-d)/4}\mathcal{G}(u). \tag{8.100}$$

We now need $\mathcal{G}(u)$ for $u > 1$. The denominator in (8.94) is $\tilde{\gamma}_d/(\sigma_d T_{\mathrm{c}}^2)$, and bearing in mind the definition (8.10) of $\mathcal{F}_C$ with $\alpha = (d-2)/2$ gives

$$\frac{\tilde{\gamma}_d}{\sigma_d T_{\mathrm{c}}^2}u\mathcal{G}(u) = \int \mathrm{d}w\, w^{(d-6)/2}\mathcal{F}_C^2(w)\,\mathrm{e}^{-2(u-1)w} \tag{8.101}$$

$$= 4\int \mathrm{d}w\, w^{(d-2)/2}\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}y'(yy')^{(d-2)/2}\,\mathrm{e}^{-2w(1-y-y'+u)} \tag{8.102}$$

$$= 2^{(4-d)/2}\Gamma\left(\frac{d}{2}\right)\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}y'(yy')^{(d-2)/2}(1-y-y'+u)^{-d/2}. \tag{8.103}$$

We now insert this into (8.100) and simplify the numerical prefactors by using $\mu_d = \lambda_d^{-1}(d-2)(4-d)/4$ and the explicit expression (A.2) for λ_d to get

$$U = \frac{T_{\mathrm{c}}}{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{4-d}{2}\right)}\left[\int_1^{\infty} \mathrm{d}u\,\mathcal{F}_M(xu/y_{\mathrm{w}})u^{-(d+6)/4}\right.$$

$$\times \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y'(yy')^{(d-2)/2}(1-y-y'+u)^{-d/2}$$
$$+ \int_1^{x/y_\mathrm{w}} \mathrm{d}u\, \mathcal{F}_M(x/uy_\mathrm{w})u^{(2-d)/4} \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y' \ldots \Bigg]. \tag{8.104}$$

This is the u-integral from (8.98) and so overall we have a four-dimensional integral over y_w, u, y, y' for $C_\mathrm{m}^{(2)}$. In general, this cannot be evaluated in closed form; a series expansion is given in [22]. The large-x behaviour, which will give us the asymptotic FDR, is easier to extract. In the first u-integral of (8.104), one can directly use the asymptotic form of $\mathcal{F}_M(xu/y_\mathrm{w})$. One can show that for large x the same replacement can be made in the second integral, and the upper integration limit sent to infinity thereafter. This gives for the large-x behaviour of U

$$U = \frac{d-2}{2}\frac{T_\mathrm{c}V_d}{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{4-d}{2}\right)}x^{(2-d)/4}y_\mathrm{w}^{(d-2)/4} \tag{8.105}$$

where V_d is a d-dependent numerical constant given by

$$V_d = \int_1^\infty \mathrm{d}u(u^{-(d+2)/2}+1)\int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y'(yy')^{(d-2)/2}(1-y-y'+u)^{-d/2}. \tag{8.106}$$

Inserting (8.105) into (8.98), the remaining y_w-integral can be done explicitly to give

$$C_\mathrm{m}^{(2)}(t,t_\mathrm{w}) = \left(\frac{d-2}{2}\right)^2 \frac{\Gamma\left(\frac{d+4}{2}\right)V_d}{\Gamma(d+1)\Gamma\left(\frac{4-d}{2}\right)}T_\mathrm{c}t_\mathrm{w}x^{-(d+2)/4}. \tag{8.107}$$

As anticipated, this has the same scaling as the first contribution (8.93) to the correlation function, so that overall for large x

$$C_\mathrm{m}(t,t_\mathrm{w}) = C_\mathrm{m}^{(1)} + C_\mathrm{m}^{(2)} = \frac{d-2}{2}T_\mathrm{c}t_\mathrm{w}x^{-(d+2)/4}\left[\frac{4}{d+2}\left(1-\frac{d-2}{2}\frac{\Gamma\left(\frac{d+4}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(d+1)}\right)\right.$$
$$\left. + \frac{d-2}{2}\frac{\Gamma\left(\frac{d+4}{2}\right)V_d}{\Gamma(d+1)\Gamma\left(\frac{4-d}{2}\right)}\right]. \tag{8.108}$$

The magnetization *response* function is rather easier to find, by using (8.14) and (8.85) in (8.39) and rescaling the integration variable to $y = t'/t_\mathrm{w}$ as usual:

$$R_\mathrm{m}(t,t_\mathrm{w}) = \int_1^x \mathrm{d}y[\delta(y-x) - x^{-1}\mathcal{F}_M(x/y)]y^{-(d-2)/4}. \tag{8.109}$$

The structure of this is rather similar to $C_\mathrm{m}^{(1)}$, and by subtracting the vanishing term (8.90) one again gets a significant cancellation,

$$R_\mathrm{m}(t,t_\mathrm{w}) = \frac{1}{x}\int_0^1 \mathrm{d}y\, \mathcal{F}_M(x/y)y^{(2-d)/4}. \tag{8.110}$$

Inserting the explicit form (8.86) of $\mathcal{F}_M$, one then gets simply

$$R_\mathrm{m}(t,t_\mathrm{w}) = x^{(2-d)/4}\left[1-\left(1-\frac{1}{x}\right)^{(d-2)/2}\right] \tag{8.111}$$

which for $x \to \infty$ behaves as

$$R_\mathrm{m}(t,t_\mathrm{w}) = \frac{d-2}{2}x^{-(2+d)/4}. \tag{8.112}$$

With expressions (8.108) and (8.112) for correlation and response in the limit of long, well-separated times, we can now finally compute the asymptotic FDR defined by

$$X_{\mathrm{m}}^{\infty} = \lim_{t \gg t_{\mathrm{w}} \gg 1} \frac{T_{\mathrm{c}} R_{\mathrm{m}}(t, t_{\mathrm{w}})}{C'_{\mathrm{m}}(t, t_{\mathrm{w}})} = \frac{4}{d+6} \left[\frac{4}{d+2} \left(1 - \frac{d-2}{2} \frac{\Gamma\left(\frac{d+4}{2}\right) \Gamma\left(\frac{d-2}{2}\right)}{\Gamma(d+1)} \right) + \frac{d-2}{2} \frac{\Gamma\left(\frac{d+4}{2}\right) V_d}{\Gamma(d+1) \Gamma\left(\frac{4-d}{2}\right)} \right]^{-1} \tag{8.113}$$

where V_d is given by (8.106) and the prefactor $4/(d+6)$ accounts for the fact that $C_{\mathrm{m}} \sim t_{\mathrm{w}} x^{-(2+d)/4} \sim t_{\mathrm{w}}^{(d+6)/4}$ and hence $C'_{\mathrm{m}}(t, t_{\mathrm{w}}) = [(d+6)/4t_{\mathrm{w}}] C_{\mathrm{m}}(t, t_{\mathrm{w}})$.

Before exploring this result, let us briefly comment on the limit as $d \to 4$, which should make contact with our results in the previous subsection. The contribution from $C_{\mathrm{m}}^{(2)}$, which appears in the second line of (8.108) and (8.113), vanishes linearly in $4-d$ in this limit because of the factor $\Gamma^{-1}((4-d)/2)$; V_d stays finite as we show below. This is consistent with the fact that in dimension $d > 4$ this term does not contribute. In the limit $d \to 4$ one has, from (8.108), $C_{\mathrm{m}} = T_{\mathrm{c}} t_{\mathrm{w}} x^{-3/2}/2$ for large x which matches precisely (8.70) for $d > 4$. Similarly, the large-x magnetization response (8.111) for $d \to 4$ is $R_{\mathrm{m}} = x^{-3/2}$ in agreement with (8.72).

We now look in more detail at the d-dependence of the asymptotic FDR (8.113) for the magnetization. Expanding in $\epsilon = 4 - d$, one has

$$\begin{aligned} X_{\mathrm{m}}^{\infty} &= \left(\frac{2}{5} + \frac{\epsilon}{25} + \mathcal{O}(\epsilon^2) \right) \left[\left(\frac{2}{3} + \frac{\epsilon}{9} \right) \left(\frac{3}{4} - \frac{\epsilon}{6} \right) + \frac{\epsilon}{8} V_4 + \mathcal{O}(\epsilon^2) \right]^{-1} \\ &= \frac{4}{5} + \left(\frac{28}{225} - \frac{V_4}{5} \right) \epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{8.114}$$

where V_4 is the limiting value of V_d for $d \to 4$, which can be worked out explicitly as

$$V_4 = -\frac{2}{3} \ln 2 + \frac{11}{12} + \frac{\pi^2}{24} \tag{8.115}$$

so that

$$X_{\mathrm{m}}^{\infty} = \frac{4}{5} + \left(\frac{2 \ln 2}{15} - \frac{53}{900} - \frac{\pi^2}{120} \right) \epsilon + \mathcal{O}(\epsilon^2). \tag{8.116}$$

It is remarkable that in a system as simple as the spherical model, where standard critical exponents are rational functions of the dimension d, the magnetization FDR for magnetized initial states is very much more complicated, and irrational already *to first order* in ϵ.

In the opposite limit $d \to 2$, the u-integral in the definition (8.106) of V_d diverges at the upper end; dropping all non-divergent corrections gives the leading divergence as

$$V_d \approx \int_1^{\infty} \mathrm{d}u \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y' (1 - y - y' + u)^{-d/2} \approx \int_1^{\infty} \mathrm{d}u \, u^{-d/2} = \frac{2}{d-2}. \tag{8.117}$$

This divergence balances the vanishing prefactor $(d-2)/2$ in (8.108) and (8.113) whereas the contribution in the round brackets coming from $C_{\mathrm{m}}^{(1)}$ vanishes linearly with $d-2$. Consequently, the asymptotic behaviour of the correlation function is, for d close to 2, $C_{\mathrm{m}} = [(d-2)/2] T_{\mathrm{c}} t_{\mathrm{w}} x^{-1}$ and the FDR becomes $X_{\mathrm{m}}^{\infty} = 4/(d+6) = 1/2$. One can also obtain the leading-order correction in $\epsilon' = d - 2$, which is given by

$$X_{\mathrm{m}}^{\infty} = \frac{1}{2} + \left(\frac{1}{16} + \frac{\pi^2}{48} \right) \epsilon' + \mathcal{O}(\epsilon'^2). \tag{8.118}$$

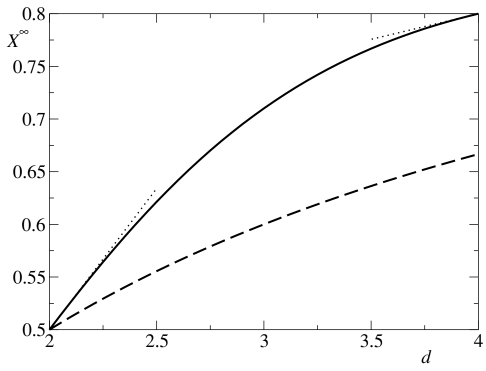

Figure 4. Asymptotic FDR X_{m}^{∞} for the magnetization, for critical coarsening with nonzero initial magnetization. Solid line: full theory (8.113) including non-Gaussian corrections; dotted lines indicate the first-order expansions near $d = 2$ and 4. Dashed line: Gaussian theory (8.12).

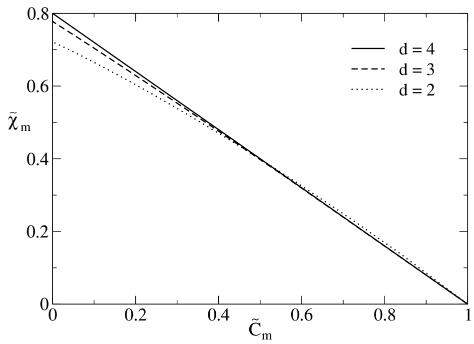

Figure 5. Normalized magnetization FD plot for $d = 2, 3, 4$, showing normalized susceptibility $\tilde{\chi}_{\mathrm{m}}$ versus normalized correlation $\tilde{C}_{\mathrm{m}}$ in the limit of long times. For $d = 4$, the plot is a straight line with (negative) slope $4/5$ as expected. Increasing deviations from this appear as d decreases towards 2.

The only subtlety here is working out the subleading term of V_d. Setting $V_d = 2/\epsilon' + a_0 + \cdots$, a_0 can be obtained as the limit for $d \to 2$ of (8.106) with $u^{-d/2}$ subtracted from the integrand. The limit can be taken in the integrand itself for finite u, giving a convergent integral with value $3/2 - \pi^2/12$. But one has to account separately for the large-u tail $u^{-d/2}\left[-1 + \int \mathrm{d}y\,\mathrm{d}y'(yy')^{(d-2)/2}\right]$ which integrates to $(-1 + 4/d^2)[2/(d-2)] \to -2$ for $d \to 2$, giving $a_0 = -1/2 - \pi^2/12$ overall.

In summary, the asymptotic magnetization FDR X_{m}^{∞} decays from $4/5$ in $d = 4$ to $1/2$ in $d = 2$. Figure 4 also shows numerical values for intermediate d. X_{m}^{∞} is larger than the FDR (8.12) from the Gaussian theory except in the limit $d \to 2$; it also remains larger throughout than the FDR (3.3) for unmagnetized initial states, shown in figure 3.

We next turn to the shape of the FD plot for the magnetization. As explained in the introduction, this is obtained by plotting the normalized susceptibility $\tilde{\chi}_{\mathrm{m}}(t, t_{\mathrm{w}}) = T_{\mathrm{c}}\chi_{\mathrm{m}}(t, t_{\mathrm{w}})/C_{\mathrm{m}}(t, t)$ versus $\tilde{C}_{\mathrm{m}}(t, t_{\mathrm{w}}) = C_{\mathrm{m}}(t, t_{\mathrm{w}})/C_{\mathrm{m}}(t, t)$. In the limit $d \to 4$, the FD plot must be a straight line with slope $X_{\mathrm{m}} = 4/5$, by continuity with the results for $d > 4$. Numerical evaluation (see figure 5) shows that, as the dimensionality decreases, the FD plots deviate progressively from this straight line. The most extreme case is the limit $d \to 2$, where analytical forms can be found.

To find the correlation function for $d \to 2$, it is useful to note that the scaling function $\mathcal{F}_M(x)$ becomes equal to $\delta(x-1)$ in the limit. Formally, one easily sees from (8.86) that for any smooth bounded function $f(x)$ and fixed $c > 1$, $\int_1^c \mathrm{d}x\, \mathcal{F}_M(x) f(x) \to f(1)$ because the divergence of $\mathcal{F}_M(x)$ at $x = 1$ becomes non-integrable in $d = 2$. We now exploit this to simplify U from (8.104). In the first term in square brackets, the argument xu/y_w of $\mathcal{F}_M$ is always larger than x/y_w and hence than 1 (except at the irrelevant boundary $y_\mathrm{w} = x$ of the y_w-integral); in the limit $d \to 2$, this contribution therefore vanishes. In the second term, replacing $\mathcal{F}_M(x/uy_\mathrm{w})$ by $\delta(x/uy_\mathrm{w} - 1)$ and expanding the prefactor to leading order in $(d-2)/2$ gives

$$\frac{U}{T_\mathrm{c}} = \frac{d-2}{2}\frac{x}{y_\mathrm{w}} \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y' (1 - y - y' + x/y_\mathrm{w})^{-1}. \tag{8.119}$$

Performing the integrals over y and y', one has

$$\frac{U}{T_\mathrm{c}} = \frac{d-2}{2}\frac{x}{y_\mathrm{w}} \left[\frac{x}{y_\mathrm{w}} \ln\left(1 - \frac{y_\mathrm{w}^2}{x^2}\right) + \ln\left(\frac{x + y_\mathrm{w}}{x - y_\mathrm{w}}\right) \right]. \tag{8.120}$$

We can now use this limit form of U to get the contribution $C_\mathrm{m}^{(2)}$ to the correlation function; recall that U was defined as the u-integral in (8.98). In the remaining y_w-integral in this equation, one can again replace $\mathcal{F}_M(1/y_\mathrm{w})$ by $\delta(1/y_\mathrm{w} - 1)$ so that only $y_\mathrm{w} = 1$ contributes, giving

$$C_\mathrm{m} = C_\mathrm{m}^{(2)} = \frac{d-2}{2} T_\mathrm{c} t_\mathrm{w} \left[x \ln\left(1 - \frac{1}{x^2}\right) + \ln\left(\frac{x+1}{x-1}\right) \right]. \tag{8.121}$$

One can show that the other contribution $C_\mathrm{m}^{(1)}$ to the magnetization correlator, given by (8.91), vanishes as $\sim(d-2)^2$ for $d \to 2$ so that to leading order $C_\mathrm{m} = C_\mathrm{m}^{(2)}$ as anticipated in writing (8.121). The corresponding response function is found by expanding (8.111) to leading order in $d - 2$:

$$R_\mathrm{m} = \frac{d-2}{2} \ln\left(\frac{x}{x-1}\right). \tag{8.122}$$

To get the FDR, it only remains to work out the t_w-derivative of C_m:

$$C_\mathrm{m}' \equiv \partial_{t_\mathrm{w}} C_\mathrm{m} = \frac{d-2}{2} T_\mathrm{c} \ln\left(\frac{x+1}{x-1}\right). \tag{8.123}$$

We therefore get, for the full dependence of the limiting FDR for $d \to 2$ on scaled time $x = t/t_\mathrm{w}$,

$$X_\mathrm{m}(x) = \frac{T_\mathrm{c} R_\mathrm{m}}{C_\mathrm{m}'} = \ln\left(\frac{x}{x-1}\right) \left[\ln\left(\frac{x+1}{x-1}\right) \right]^{-1}. \tag{8.124}$$

For $x \to \infty$, this gives $X_\mathrm{m}^\infty = 1/2$ consistent with the discussion above. In the limit $x \to 1$ of comparable times, on the other hand, $X_\mathrm{m}(x)$ approaches 1, logarithmically slowly; the FD plot for $d \to 2$ therefore starts off with a pseudo-equilibrium slope. Interestingly, this implies that the trends of the slope with d are different at the two ends of the plot: for well-separated times ($x \to \infty$), the slope *decreases* from 4/5 to 1/2 as d decreases from 4 to 2; for comparable times ($x \to 1$), it *increases* from 4/5 to 1.

To get the FD plot itself we need the susceptibility, which is found by integration of the response (8.111) as

$$\chi_{\mathrm{m}}(t,t_{\mathrm{w}}) = \int_{t_{\mathrm{w}}}^{t} \mathrm{d}t'\, R_{\mathrm{m}}(t/t') = t\int_{1/x}^{1} \mathrm{d}z\, R_{\mathrm{m}}(1/z) \tag{8.125}$$

$$= t\int_{1/x}^{1} \mathrm{d}z\, z^{(d-2)/4}[1-(1-z)^{(d-2)/2}]. \tag{8.126}$$

Expanding to linear order in $d-2$ and integrating gives

$$\chi_{\mathrm{m}}(t,t_{\mathrm{w}}) = \frac{d-2}{2}t\frac{x-1}{x}\left[1-\ln\left(\frac{x-1}{x}\right)\right]. \tag{8.127}$$

The normalized susceptibility $\tilde{\chi}_{\mathrm{m}}$ is obtained by dividing by $C_{\mathrm{m}}(t,t)/T_{\mathrm{c}}$, which from (8.121) for $x \to 1$ equals $2\ln 2[(d-2)/2]t$. The resulting FD plot for $d \to 2$ is shown in figure 5, together with the ones for $d = 3$ (determined numerically) and $d = 4$. For $d \to 2$, the approach of the slope to the equilibrium value $X = 1$ for $x \to 1$ is difficult to see because it is logarithmically slow. As expected from the trends with d in the initial and final slopes of the FD plot, the curve for $d \to 2$ is the most strongly curved, while in $d = 4$ we have the anticipated straight line with slope 4/5 that is required by continuity with the results for $d > 4$.

To quantify the shape of the FD plot further one can also consider the axis ratio Y, defined as the limiting value of $\tilde{\chi}_{\mathrm{m}}(t,t_{\mathrm{w}}) = T_{\mathrm{c}}\chi_{\mathrm{m}}(t,t_{\mathrm{w}})/C_{\mathrm{m}}(t,t)$ for well-separated times $t \gg t_{\mathrm{w}} \gg 1$. In equilibrium, this would correspond to the FDT for the static quantities, i.e. equal-time fluctuations and static susceptibilities. Out of equilibrium, if the FD plot is straight then Y coincides with X; if it is not, then is has been argued that in some circumstances Y can be more relevant for characterizing effective temperatures than X [23]. For the magnetization FD plot in the current scenario of magnetized initial states, we see from figure 5 that Y decreases along with X_{m}^{∞} from 4/5 as the dimension is lowered below $d = 4$. The two quantities begin to differ more noticeably as d decreases further, with $X_{\mathrm{m}}^{\infty} = 1/2$ and $Y = 1/(2\ln 2) = 0.7213\ldots$ in the limit $d \to 2$.

9. Summary and discussion

In this paper, we have considered the non-equilibrium dynamics of the spherical ferromagnet after a quench to the critical temperature T_{c}. Our focus has been the calculation of correlation and response functions and the associated fluctuation–dissipation ratios (FDRs) $X(t,t_{\mathrm{w}})$. The key quantity that can be extracted from the latter is the asymptotic FDR X^{∞} for large and well-separated times $t \gg t_{\mathrm{w}} \gg 1$; it is independent of model-specific details within a given dynamical universality class. We were motivated by two questions: how does X^{∞} depend on the observable considered, both with regard to the length scale and the type of observable (spin, bond, spin product)? And what is the effect of initial conditions, in particular the presence of a nonzero magnetization in the initial state? The first question has implications for the interpretation of T/X^{∞} as an effective temperature, which is plausible only if this quantity is observable independent. The second one allowed us to uncover whether different initial conditions can lead to different universality classes of critical coarsening.

A peculiarity of the spherical model is the weak infinite-range interaction produced by the spherical constraint. This requires that one distinguishes between long-range or 'block' observables, which probe length scales large compared to the (time-dependent) correlation length but small compared to the system size, and global observables whose behaviour depends on correlations across the entire system. Technically, the first case is much easier to treat because the standard theory where the spins have Gaussian statistics can be used. Global correlation and response functions, on the other hand, require non-Gaussian corrections arising from the fluctuations of the effective Lagrange multiplier.

We dealt with the case of finite-range (i.e., either local or long-range) observables in section 3. For spin observables, we found in the long-range case (3.3) the same X^∞ as for local spin correlations and response [9]. This was as expected from the general correspondence between local and long-range observables discussed in the introduction. The FD plot for the long-range spin observable, i.e. the magnetization, is a straight line in the long-time limit. This is as in the Ising case in $d = 1$ [10], but it is interesting to note that here it holds for all dimensions $d > 2$. On the other hand, in the Ising model with $d \geqslant 2$, RG arguments have been adduced [11, 15, 17] to suggest that the magnetization FD plot should not be straight, though with deviations that are likely too small to be detectable numerically [15]. It is likely that the Gaussian statistics of the spherical model are responsible for producing a simpler, straight-line magnetization FD plot in all dimensions, although it would be interesting to know whether any other models have this property.

We then looked at the effect of the type of observable on X^∞, considering both bond and spin product observables in either the local or long-range versions. The results in equations (3.15), (3.20), (3.26), (3.31) show that, although the precise time dependence of $X(t, t_w)$ varies, the asymptotic FDR X^∞ is the same in all cases. This is consistent with general arguments [13] suggesting that for a Gaussian theory all observables should yield the same X^∞. In contrast to the Ising case [10], not all long-range observables give nontrivial FD plots; in fact, only the block product observable does so, and only for $d < 4$, while all others produce pseudo-equilibrium FD plots for long times.

The bulk of the paper was concerned with the more challenging analysis of global correlation and response functions, focusing mostly on the energy as a key observable. In section 4, we constructed a framework for calculating non-Gaussian corrections to the spins, which are $\mathcal{O}(N^{-1/2})$ to leading order. This led to the general expression (4.11) for these leading-order corrections. It involves a two-time kernel $L(t, t_w)$ which from (4.9) is the functional inverse of $K(t, t_w)$ defined in (4.8). The basis of all subsequent calculations is the determination of the long-time scaling of these two functions, as summarized at the end of section 4.2.

In section 5, we obtained general expressions for energy correlation and response functions, in terms of the kernel L and other quantities known from the Gaussian theory; the results can be found in equations (5.5), (5.12), (5.14), (5.19) and (5.29). Evaluating these first for the equilibrium case, we found that the energy correlation and susceptibility display a plateau for T just below T_c and $d > 4$; this is caused by the $\mathbf{q} = \mathbf{0}$ wavevector, i.e. by the slow relaxation of the global magnetization. In section 6, we proceeded to the long-time analysis of energy FD behaviour in the non-equilibrium case for $d > 4$; the key results are (6.23) and (6.24). The associated FDR is given explicitly in (6.22) and has the *same* asymptotic value $X^\infty = 1/2$ as for all other (finite-range) observables in $d > 4$. The analysis of the case $d < 4$ is more difficult, and we were able to find closed-form results (7.15), (7.16) only in the limit of well-separated times $t/t_w \gg 1$. This is, however, enough to determine X^∞, with the result (7.19). Evaluating this, both numerically and by expansion in $4 - d$ and $d - 2$, the crucial conclusion is that it does not coincide with the asymptotic FDR for finite-range observables, see figure 3. A naive interpretation of T/X^∞ as an effective temperature for critical coarsening dynamics is therefore ruled out, since such a temperature ought to be observable independent. On the other hand, to first order in $4 - d$ the result agrees with an RG calculation [13] for the $O(n)$-model. We conclude that non-Gaussian corrections to the FD behaviour of global observables in the spherical model capture genuine physical effects that have close counterparts in more realistic systems with only short-range interactions.

Finally, in section 8 we turned our attention to critical coarsening starting from magnetized initial states; physically, this situation could be produced by an up-quench from an equilibrated state at a starting temperature $T < T_c$. We concentrated on the simpler spin observables and found that already for them the presence of a nonzero magnetization makes global properties sensitive to non-Gaussian corrections. As with the energy fluctuations, it is the *global* correlation and response functions that make contact with the results for short-range models, as obtained recently for the Ising case [20]: we find $X_m^\infty = 4/5$ for $d > 4$, equation (8.73). This is distinct from the value $X^\infty = 1/2$ for the unmagnetized case, indicating that magnetized critical coarsening is in a separate dynamical universality class. Surprisingly, even the expressions for correlation and response functions themselves, which are not expected to be universal, coincide with those for the Ising case. It remains to be understood whether this is accidental or has more profound origins. For the case $d < 4$, we obtained new *exact* values for the asymptotic FDR of magnetized critical coarsening. The magnetization response (8.111) can be found explicitly for long times, while for the magnetization correlator only the asymptotics for well-separated times (8.108) can be written in closed form. The resulting X_m^∞, equation (8.113), is surprisingly nontrivial: while it matches continuously with $X_m^\infty = 4/5$ in $d > 4$ and approaches the simple value $X_m^\infty = 1/2$ for $d \to 2$ as shown in figure 4, it is irrational already to first order in an expansion in $4 - d$, equation (8.116), or $d - 2$, equation (8.118).

While the conclusion of our calculation as regards the existence of a well-defined effective temperature for critical coarsening is negative, the issue of dynamic universality classes and new asymptotic FDRs due to magnetized (and possibly other, different) initial conditions clearly deserves further study. Results for systems with short-range interactions, such as the $O(n)$ and n-vector models, would be particularly welcomed. After the present work was completed, we became aware that a first step in this direction has recently been taken by the authors of [24], who calculated the FDR for the n-vector model with a magnetized initial state within an ϵ-expansion around $d = 2$. Because of the vectorial nature of the order parameter one has to distinguish between longitudinal and transverse correlation and response in this case. The *longitudinal* quantities are most closely analogous to those we calculated in the spherical model and correspondingly give $X^\infty = 1/2$ for $d = 2$ [24]. It would clearly be desirable to calculate corrections in $d - 2$ to compare in more detail with our result (8.118). This could clarify if there are any genuine differences between the spherical and n-vector (with $n \to \infty$) models, which are known to have identical properties in equilibrium [25] and within a Gaussian theory of the dynamics. The *transverse* FDR of the n-vector model has a first-order correction in $d - 2$ that remains rational even for $n \to \infty$ [24] and so necessarily differs from (8.118) and, presumably, the longitudinal FDR. It remains an open problem whether analogues of transverse quantities might exist in appropriately modified spherical models.

As regards future work, we first note that a complete classification of dynamical universality classes within critical coarsening remains to be achieved. An earlier study of the spherical model considered initial conditions with long-range correlations but no overall magnetization; this yields no new (nonzero) values of the asymptotic FDR X^∞ [26]. The presence of a nonzero magnetization thus appears to be important for observing new phenomena and is reflected in our calculation by the fact that non-Gaussian fluctuations become important. Whether there are yet other initial conditions that could give rise to distinct values of X^∞ is an open problem.

Our general framework for treating non-Gaussian corrections to the dynamics can also be applied in other contexts. For example, it can be used to analyse the *fluctuations* across thermal histories of correlation and response functions that have been coarse-grained across a finite-sized system. The properties of these fluctuations should be useful for understanding dynamical heterogeneities in coarsening dynamics [27], and we will report on the results of such a study shortly. We have also extended our approach to non-Gaussian corrections for the dynamics of, e.g., the $O(n)$-model with large but finite n, opening up the attractive prospect of obtaining exact results analogous to the ones in this paper for models with exclusively short-range interactions. Finally, as this manuscript went to press we discovered an exact solution for the inverse kernel $L(t,t')$ which should allow one to further extend the analysis to include, for example, initial-slip effects on FDRs in magnetized coarsening.

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Appendix A. Evaluation of X_E^∞

In this appendix, we evaluate the various numerical factors in the asymptotic FDR (7.18) for the energy in $d<4$. For β_d, we already have an expression (7.14). By definition, γ_d is the large-t limit of $CC(t,t)/t^{(4-d)/2}$. Its value can be deduced from (7.3): for small q, $q=\omega^2$ and hence $(\mathrm{d}q)=\sigma_d\,\mathrm{d}\omega\,\omega^{(d-2)/2}$. (The value of the proportionality constant, $\sigma_d=(4\pi)^{-d/2}\Gamma^{-1}(d/2)$, is not actually needed explicitly because it cancels from the overall result.) Thus, from (7.3)

$$\gamma_d=\sigma_d T_c^2\int \mathrm{d}w\,w^{(d-6)/2}\mathcal{F}_C^2(w). \tag{A.1}$$

Next, λ_d is defined by $L_{\mathrm{eq}}^{(2)}(t)=\lambda_d t^{(d-6)/2}$ for large t. The Laplace transform of $L_{\mathrm{eq}}^{(2)}$ therefore has a singular term $\hat{L}_{\mathrm{eq}}^{(2)}(s)=\lambda_d\Gamma((d-4)/2)s^{(4-d)/2}$ for $s\to 0$. From (4.21), this must match the corresponding singularity in $-1/\hat{K}_{\mathrm{eq}}(s)$. The latter follows from (4.22) as $\hat{K}_{\mathrm{eq}}(s)=s^{(d-4)/2}T_c\sigma_d\int \mathrm{d}w\,w^{(d-4)/2}(1+2w)^{-1}=s^{(d-4)/2}T_c\sigma_d 2^{(2-d)/2}\Gamma((4-d)/2)\Gamma((d-2)/2)$. This tells us that

$$\lambda_d^{-1}=-T_c\sigma_d 2^{(2-d)/2}\Gamma\left(\frac{4-d}{2}\right)\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d-2}{2}\right). \tag{A.2}$$

Finally, we need α_d from (7.13). Inserting (2.25) into (7.7) and using (A.1), the scaling function $\mathcal{G}(x)$ can be written for $x\geqslant 1$ as

$$\mathcal{G}(x)=\frac{4\sigma_d T_c^2}{\gamma_d}\int \mathrm{d}w\,w^{(d-6)/2}w^2\int_0^1\mathrm{d}y\int_0^1\mathrm{d}y'(yy')^{(d-4)/2}\,\mathrm{e}^{-2(1+x-y-y')w} \tag{A.3}$$

$$=\frac{\Gamma\left(\frac{d}{2}\right)2^{(4-d)/2}\sigma_d T_c^2}{\gamma_d}\int_0^1\mathrm{d}y\,y^{(d-4)/2}\int_0^1\frac{\mathrm{d}y'}{y'^2}[(1+x-y)/y'-1]^{-d/2} \tag{A.4}$$

$$=\frac{2}{d-2}\frac{\Gamma\left(\frac{d}{2}\right)2^{(4-d)/2}\sigma_d T_c^2}{\gamma_d}\int_0^1\mathrm{d}y\,y^{(d-4)/2}\frac{(x-y)^{(2-d)/2}}{1+x-y}. \tag{A.5}$$

We define the integral as $\tilde{\mathcal{G}}(x)$. It is an unnormalized version of $\mathcal{G}(x)$; because of $\mathcal{G}(1)=1$, one then has $\mathcal{G}(x)=\tilde{\mathcal{G}}(x)/\tilde{\mathcal{G}}(1)$. The corresponding unnormalized value of α_d, $\tilde{\alpha}_d=\alpha_d\tilde{\mathcal{G}}(1)$, is

$$\tilde{\alpha}_d = \int_0^\infty \mathrm{d}x\, x[\tilde{g}_d x^{-d/2} - \tilde{\mathcal{G}}(x)] \tag{A.6}$$

$$= \frac{2\tilde{g}_d}{4-d} - \int_0^1 \mathrm{d}x\, x\tilde{\mathcal{G}}(x) + \int_1^\infty \mathrm{d}x\, x[\tilde{g}_d x^{-d/2} - \tilde{\mathcal{G}}(x)] \tag{A.7}$$

$$= \frac{4}{(4-d)(d-2)} - \int_1^\infty \mathrm{d}x\, x^{-(d+2)/2}\tilde{\mathcal{G}}(x) + \int_1^\infty \mathrm{d}x\, x\left[\frac{2x^{-d/2}}{d-2} - \tilde{\mathcal{G}}(x)\right]. \tag{A.8}$$

Here, we have used (7.7) to express the values of $\tilde{\mathcal{G}}(x)$ for $x < 1$ in terms of those for $x > 1$; we also inserted $\tilde{g}_d = g_d\tilde{\mathcal{G}}(1) = \lim_{x\to\infty}\tilde{\mathcal{G}}(x)x^{d/2} = 2/(d-2)$ which follows from (A.5). The first integral has no divergences. The second one could in principle be left as it is for numerical evaluation, but it is useful to rewrite it using a dimensional regularization trick, as follows. If we extend the definition of $\tilde{\mathcal{G}}(x)$ to $d > 4$ using (A.5), then both parts of the second integral in (A.8) are separately finite for $4 < d < 6$, and we can evaluate them first in that range of d and then analytically continue to $d < 4$. (The region $x \approx 1$ causes no difficulty since $\tilde{\mathcal{G}}(x) \sim (x-1)^{(4-d)/2}$ for $x \to 1$, which remains integrable for $d < 6$.) The first part gives $[2/(d-2)][2/(d-4)]$ and just cancels the first term in (A.8), so that

$$\int_1^\infty \mathrm{d}x\, x\left[\frac{2x^{-d/2}}{d-2} - \tilde{\mathcal{G}}(x)\right] + \frac{4}{(4-d)(d-2)}$$

$$= -\int_1^\infty \mathrm{d}x \int_0^1 \mathrm{d}y\, y^{(d-4)/2}(x-y)^{(2-d)/2}\frac{x}{1+x-y} \tag{A.9}$$

$$= -\frac{\pi}{\sin[\pi(d-4)/2]} + \int_1^\infty \mathrm{d}x \int_0^1 \mathrm{d}y\, y^{(d-4)/2}\frac{(x-y)^{(2-d)/2}(1-y)}{1+x-y}. \tag{A.10}$$

The cancellation of the pure power-law term proportional to $\tilde{g}_d$ from the integral (7.13) is a feature well known from dimensional regularization in field theory. Inserting the last expression into (A.8) then gives (7.20) in the main text.

Collecting the above results, the asymptotic FDR (7.18) for the energy in $d < 4$ is

$$X_E^\infty = \frac{4T_\mathrm{c}}{d\tilde{\alpha}_d}\left[\frac{2}{d-2}\frac{\Gamma\left(\frac{d}{2}\right)2^{(4-d)/2}\sigma_d T_\mathrm{c}^2}{\gamma_d}\left(-\frac{\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d)}\right)\gamma_d\right]^{-1}$$

$$\times\left[-T_\mathrm{c}\sigma_d 2^{(2-d)/2}\Gamma\left(\frac{4-d}{2}\right)\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d-2}{2}\right)\right] \tag{A.11}$$

$$= \frac{2}{d\tilde{\alpha}_d}\frac{\Gamma(d)\Gamma\left(\frac{4-d}{2}\right)}{\Gamma\left(\frac{d+4}{2}\right)} \tag{A.12}$$

which is the result stated in (7.19).

Appendix B. Solution for $\mathcal{F}_L(x)$ for magnetized case in $d < 4$

In this appendix, we prove that the solution $\mathcal{F}_L(x) = 2/(4-d)x^{(2-d)/2} + (2-d)/(4-d)x^{-d/2}$ given in (8.81) does indeed satisfy the integral equation (4.34)

$$\int_1^x \mathrm{d}y(x-y)^{(2-d)/2}(y-1)^{(d-6)/2}[\mathcal{F}_K(x/y)\mathcal{F}_L(y) - \mathcal{F}_K(x)] = 0 \tag{B.1}$$

with $d < 4$ and $\mathcal{F}_K(x)$ given by (8.78) for the magnetized case considered here.

The difference in the square brackets can be written as the integral of a partial derivative

$$\mathcal{F}_K(x/y)\mathcal{F}_L(y) - \mathcal{F}_K(x) = \int_1^y \mathrm{d}z\, \partial_z[\mathcal{F}_K(x/z)\mathcal{F}_L(z)] \tag{B.2}$$

so that, after exchanging the order of the integrals, the left-hand side of (B.1) becomes

$$l = \int_1^x \mathrm{d}z\, \partial_z[\mathcal{F}_K(x/z)\mathcal{F}_L(z)] \int_z^x \mathrm{d}y (x-y)^{(2-d)/2}(y-1)^{(d-6)/2} \tag{B.3}$$

$$= \frac{1}{x-1}\frac{2}{4-d}\int_1^x \mathrm{d}z (x-z)^{(4-d)/2}(z-1)^{(d-4)/2}\partial_z[\mathcal{F}_K(x/z)\mathcal{F}_L(z)] \tag{B.4}$$

where in the second step the y-integral has been performed. Equation (B.4) can equivalently be obtained from (B.1) by integrating by parts. From (8.78) for $\mathcal{F}_K(x)$ and the form given above for $\mathcal{F}_L(x)$, we can find the z-derivative of $\mathcal{F}_K(x/z)\mathcal{F}_L(z)$ explicitly. After simplifying the latter, equation (B.4) becomes

$$l = \eta \int_1^x \mathrm{d}z (x-z)^{(4-d)/2}(z-1)^{(d-4)/2}\left\{-\frac{1}{z} + \frac{d}{2z^2} + \frac{d-2}{2}\left[\frac{d}{2}x - z(x+1)\right]\right.$$
$$\left.\times\, z^{-(d+2)/2}(x-z)^{(d-4)/2}\int_{z/x}^1 \mathrm{d}y (1-y)^{(2-d)/2} y^{(d-4)/2}\right\} \tag{B.5}$$

where

$$\eta = \frac{x^{(2-d)/2}}{x-1}\frac{2(d-2)}{(4-d)^2} \tag{B.6}$$

collects all the prefactors. By rescaling y by a factor x, the last integral in (B.5) can be transformed to $\int_z^x \mathrm{d}y (x-y)^{(2-d)/2} y^{(d-4)/2}$. Interchanging the y- and z-integrals then gives

$$\frac{l}{\eta} = \int_1^x \mathrm{d}z (x-z)^{(4-d)/2}(z-1)^{(d-4)/2}\left(-\frac{1}{z} + \frac{d}{2z^2}\right) + \int_1^x \mathrm{d}y (x-y)^{(2-d)/2} y^{(d-4)/2}$$
$$\times \int_1^y \mathrm{d}z\, z^{-(d+2)/2}(z-1)^{(d-4)/2}\frac{d-2}{2}\left[\frac{d}{2}x - z(x+1)\right]. \tag{B.7}$$

The integral on the last line can now be calculated explicitly to give

$$g(y) = \left(\frac{y-1}{y}\right)^{(d-2)/2}\left(\frac{d-2}{2}\frac{x}{y} - 1\right). \tag{B.8}$$

Relabelling y to z, equation (B.7) can thus be written as a single integral:

$$\frac{l}{\eta} = \int_1^x \mathrm{d}z (x-z)^{(2-d)/2}\left[(z-1)^{(d-4)/2}(x-z)\left(-\frac{1}{z} + \frac{d}{2z^2}\right) + z^{(d-4)/2} g(z)\right] \tag{B.9}$$

$$= \int_1^x \mathrm{d}z (x-z)^{(2-d)/2}(z-1)^{(d-4)/2}\left[\frac{x}{z^2} - \left(\frac{d-2}{2} + \frac{4-d}{2}x\right)\frac{1}{z}\right]. \tag{B.10}$$

Finally, the variable change $v = (x/z-1)/(x-1)$ transforms the integration range to $0\ldots 1$ and leads after a few simplifications to straightforward beta-function integrals:

$$\frac{l}{\eta} = x^{(2-d)/2}(x-1)\int_0^1 \mathrm{d}v\, v^{(2-d)/2}\left(-\frac{4-d}{2} + v\right)(1-v)^{(d-4)/2} \tag{B.11}$$

$$= x^{(2-d)/2}(x-1)\left[-\frac{4-d}{2}\frac{\Gamma\left(\frac{4-d}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(1)} + \frac{\Gamma\left(\frac{6-d}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(2)}\right] = 0. \tag{B.12}$$

This proves that (B.1) is indeed satisfied, as required.

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